

Finite rank perturbations of unitary operators and Clark's model in general situation

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- 1 Main objects: Finite rank perturbations and models
 - Finite rank perturbations
 - Functional models for c.n.u. contractions
 - Defects and characteristic function for T_Γ
- 2 Toward a formula for the adjoint Clark operator
 - Spectral representation of unitary perturbations
 - Model, agreement of parametrizing operators
 - Representation formula, rank 1 case
- 3 Adjoint Clark operator in general case
 - A universal formula in matrix case
 - Representation with matrix weights
 - To do list

Finite rank perturbations

- U unitary in \mathcal{H} , subset $R \subset \mathcal{H}$ is fixed, $\dim R = d$.
- Operators K , $\text{Ran } K \subset R$, such that $U + K$ is unitary are parametrized by $d \times d$ unitary matrices: namely fix unitary $\mathbf{B} : \mathbb{C}^d \rightarrow R$ then unitary $d \times d$ matrices Γ parametrize all unitary perturbed operators

$$U_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*U$$

Indeed, trivial when $U = \mathbf{I}$, and right multiplying by U get the formula.

- Familiar parametrization for rank one perturbations

$$U_\gamma = U + (\gamma - 1)bb^*U = U + (\gamma - 1)b(b_*)^*, \quad b_* = U^*b.$$

$$\|b\| = 1.$$

Finite rank perturbations

$$U_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})B^*U, \quad \Gamma : \mathbb{C}^d \rightarrow \mathbb{C}^d.$$

- WLOG $R = \text{Ran } \mathbf{B}$ is $*$ -cyclic.
- If Γ is a strict contraction, then $U_\Gamma =: T_\Gamma$ is a completely non-unitary (c.n.u.) contraction.

Use T_Γ instead of U_Γ to emphasize that it is not unitary.

- As c.n.u. T_Γ admits a functional model $\mathcal{M}_\Gamma = \mathcal{M}_{T_\Gamma}$

Goal

$$T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*U, \quad \Gamma : \mathbb{C}^d \rightarrow \mathbb{C}^d, \quad \|\Gamma\| < 1.$$

- Consider U in its spectral representation.
- We assumed that $\text{Ran } \mathbf{B}$ is $*$ -cyclic, so T_Γ is c.n.u.
- T_Γ is unitarily equivalent to its functional model $\mathcal{M}_\Gamma : \mathcal{K}_\theta \rightarrow \mathcal{K}_\theta$, (for example Sz.-Nagy-Foiaş model), where $\theta = \theta_T$ is the characteristic function.
- Want to describe the Clark operator, i.e. a unitary operator $\Phi = \Phi_\Gamma$ such that

$$T_\Gamma \Phi_\Gamma = \Phi_\Gamma \mathcal{M}_\Gamma.$$

U in spectral representation

- WLOG assume that $U = M_\xi$ in

$$\mathcal{H} = \int_{\mathbb{T}}^{\oplus} E(\xi) d\mu(\xi),$$

$E(\xi) = \text{span}\{e_k : 1 \leq k \leq N(\xi)\} \subset E$, $\{e_k\}_k$ — ONB in E .

- $\mathcal{H} \subset L^2(\mu; E)$.
- Define matrix function B , $B(\xi) : \mathbb{C}^d \rightarrow E(\xi) \subset E$,

$$B(\xi)e = \mathbf{B}e(\xi), \quad e \in \mathbb{C}^d.$$

- $\text{Ran } \mathbf{B}$ is $*$ -cyclic iff

$$\text{Ran } B(\xi) = E(\xi) \quad \mu\text{-a.e.}$$

Functional model

Let T be a c.n.u.

Defect operators and subspaces,

$$\begin{aligned} D_T &:= (\mathbf{I} - T^*T)^{1/2}, & D_{T^*} &:= (\mathbf{I} - TT^*)^{1/2}, \\ \mathfrak{D}_T &:= \text{clos Ran } D_T, & \mathfrak{D}_{T^*} &:= \text{clos Ran } D_{T^*}. \end{aligned}$$

Let $\dim \mathfrak{D} = \dim \mathfrak{D}_T$, $\dim \mathfrak{D}_* = \dim \mathfrak{D}_{T^*}$, and let

$$V : \mathfrak{D}_T \rightarrow \mathfrak{D}, \quad V_* : \mathfrak{D}_{T^*} \rightarrow \mathfrak{D}_*$$

be unitary operators (coordinate operators).

The characteristic function $\theta = \theta_T$, $\theta(z) : \mathfrak{D} \rightarrow \mathfrak{D}_*$ is defined as

$$\theta_T(z) = V_* \left(-T + zD_{T^*} (\mathbf{I}_{\mathcal{H}} - zT^*)^{-1} D_T \right) V^* \Big|_{\mathfrak{D}}, \quad z \in \mathbb{D}.$$

Functional model(s)

Following Nikolskii–Vasyunin [7] the functional model is constructed as follows:

- 1 For a contraction $T : \mathcal{K} \rightarrow \mathcal{K}$ consider its *minimal* unitary dilations $\mathcal{U} : \mathcal{H} \rightarrow \mathcal{H}$, $\mathcal{K} \subset \mathcal{H}$,

$$T^n = P_{\mathcal{K}} \mathcal{U}^n |_{\mathcal{K}}, \quad n \geq 0.$$

- 2 Pick a spectral representation of \mathcal{U}
- 3 Work out formulas in this spectral representation
- 4 Model subspace $\mathcal{K} = \mathcal{K}_\theta$ is usually a subspace of a weighted space $L^2(\mathfrak{D}_* \oplus \mathfrak{D}, W)$, $\mathfrak{D} \cong \mathfrak{D}_T$, $\mathfrak{D}_* \cong \mathfrak{D}_{T^*}$ with some operator-valued weight.

Specific representations give us a *transcription* of the model.

Among common transcriptions are: the Sz.-Nagy–Foiiaş transcription, the de Branges–Rovnyak transcription, Pavlov transcription.

Sz.-Nagy–Foiiaş and de Branges–Rovnyak transcriptions

- **Sz.-Nagy–Foiiaş:** $\mathcal{H} = L^2(\mathfrak{D}_* \oplus \mathfrak{D})$ (non-weighted, $W \equiv I$).

$$\mathcal{K}_\theta := \begin{pmatrix} H_{\mathfrak{D}_*}^2 \\ \text{clos } \Delta L_{\mathfrak{D}}^2 \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H_{\mathfrak{D}}^2,$$

where $\Delta(z) := (1 - \theta(z)^* \theta(z))^{1/2}$, $z \in \mathbb{T}$.

- **de Branges–Rovnyak:** $\mathcal{H} = L^2(\mathfrak{D}_* \oplus \mathfrak{D}, W_\theta^{[-1]})$, where

$$W_\theta(z) = \begin{pmatrix} I & \theta(z) \\ \theta(z)^* & I \end{pmatrix}$$

and $W_\theta^{[-1]}$ is the Moore–Penrose inverse of W_θ . \mathcal{K}_θ is given by

$$\left\{ \begin{pmatrix} g_+ \\ g_- \end{pmatrix} : g_+ \in H^2(\mathfrak{D}_*), g_- \in H_-^2(\mathfrak{D}), g_- - \theta^* g_+ \in \Delta L^2(\mathfrak{D}) \right\}.$$

Defects and characteristic function for T_Γ

Recall: $T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*U$, $\Gamma : \mathbb{C}^d \rightarrow \mathbb{C}^d$, $\|\Gamma\| < 1$.

- $\mathfrak{D}_{T_\Gamma} = \text{Ran}(\mathbf{B}^*U)^* = \text{Ran } U^*\mathbf{B}$ and $\mathfrak{D}_{T_\Gamma^*} = \text{Ran } \mathbf{B}$
- In the scalar case \mathfrak{D}_{T_γ} and $\mathfrak{D}_{T_\gamma^*}$ are spanned by the vectors $\bar{\xi}$ and $\mathbf{1}$ respectively.
- Characteristic function θ_T of a contraction T is defined as

$$\theta_T(z) = V_* \left(-T + zD_{T^*} (\mathbf{I}_{\mathcal{H}} - zT^*)^{-1} D_T \right) V^* \Big|_{\mathfrak{D}}, \quad z \in \mathbb{D}.$$

In our case $V_* = \mathbf{B}^*$, $V = (\mathbf{B}^*U)^* = U^*\mathbf{B}$,

$$T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*U, \quad \Gamma : \mathbb{C}^d \rightarrow \mathbb{C}^d, \quad \|\Gamma\| < 1.$$

and $(\mathbf{I} - zU^*)^{-1}$ is just the multiplication by $(1 - z\bar{\xi})^{-1}$.

- To compute it use Woodbury inversion formula:
if $B, C : E \rightarrow \mathcal{H}$ ($\dim E$ is small), then

$$(\mathbf{I}_{\mathcal{H}} - CB^*)^{-1} = \mathbf{I}_{\mathcal{H}} + C(\mathbf{I}_E - B^*C)^{-1}B^*.$$

To get this formula just decompose $(\mathbf{I}_{\mathcal{H}} - CB^*)^{-1}$ using geometric series. A formal proof can be obtained just by checking.

- In rank one case we get the Sherman–Morrison inversion formula:

$$(I - cb^*)^{-1} = I + \frac{1}{d}cb^*, \quad d = (c, b) = b^*c.$$

- $I - zT_\Gamma^*$ is a finite rank perturbation of $I - zU_1^* = I - zM_{\bar{\xi}}$;
- The inverse of $I - zM_{\bar{\xi}}$ is multiplication by $(1 - z\bar{\xi})^{-1}$, so Cauchy integrals appear.

Cauchy Transforms

- Define Cauchy integrals

$$\mathcal{C}_1\tau(z) := \int_{\mathbb{T}} \frac{\bar{\xi}z d\tau(\xi)}{1 - \bar{\xi}z}, \quad \mathcal{C}_2\tau(z) := \int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \bar{\xi}z} d\tau(\xi).$$

- Consider matrix-valued measure $B(\xi)^*B(\xi)d\mu(\xi)$ ($B^*B\mu$ as shorthand), and let

$$F_1(z) := \mathcal{C}_1[B^*B\mu](z), \quad F_2(z) := \mathcal{C}_2[B^*B\mu](z), \quad z \in \mathbb{D}$$

be the corresponding matrix-valued Cauchy transforms

Characteristic function for T_Γ

- Characteristic function θ_γ of T_γ :

$$\theta_\gamma(z) = -\gamma + \frac{(1 - |\gamma|^2)\mathcal{C}_1\mu(z)}{1 + (1 - \bar{\gamma})\mathcal{C}_1\mu(z)} = \frac{(1 - \gamma)\mathcal{C}_2\mu(z) - (1 + \gamma)}{(1 - \bar{\gamma})\mathcal{C}_2\mu(z) + (1 + \bar{\gamma})},$$

- Note that $\theta_\gamma(0) = -\gamma$, because $\mathcal{C}_1\mu(0) = 0$
- In the matrix case

$$\begin{aligned}\theta_\Gamma(z) &= -\Gamma + D_{\Gamma^*} F_1(z) \left(\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}}) F_1(z) \right)^{-1} D_\Gamma \\ &= -\Gamma + D_{\Gamma^*} \left(\mathbf{I}_{\mathfrak{D}} - F_1(z) (\Gamma^* - \mathbf{I}_{\mathfrak{D}}) \right)^{-1} F_1(z) D_\Gamma,\end{aligned}$$

Characteristic function for T_0

- For $\gamma = 0$

$$\theta_0(z) = \frac{R_1\mu(z)}{1 + R_1\mu(z)} = \frac{R_2\mu(z) - 1}{R_2\mu(z) + 1}, \quad z \in \mathbb{D}.$$

- For $\Gamma = \mathbf{0}$

$$\begin{aligned}\theta_{\mathbf{0}}(z) &= F_1(z)(\mathbf{I} + F_1(z))^{-1} = (\mathbf{I} + F_1(z))^{-1}F_1(z) \\ &= (F_2(z) - \mathbf{I})(F_2(z) + \mathbf{I})^{-1} = (F_2(z) + \mathbf{I})^{-1}(F_2(z) - \mathbf{I}).\end{aligned}$$

LFTs for characteristic functions

- In the scalar case

$$\theta_\gamma(z) = \frac{\theta_0(z) - \gamma}{1 - \bar{\gamma}\theta_0(z)},$$

- In the matrix case

$$\begin{aligned}\theta_\Gamma &= D_{\Gamma^*}^{-1}(\theta_0 - \Gamma)(\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_0)^{-1}D_\Gamma \\ &= D_{\Gamma^*}(\mathbf{I}_{\mathfrak{D}} - \theta_0\Gamma^*)^{-1}(\theta_0 - \Gamma)D_\Gamma^{-1}\end{aligned}$$

“Model” case of unitary perturbations

Recall: $U_\alpha = U_1 + (\alpha - 1)b(b_*)^*$, $|\alpha| = 1$

$$U_1 = M_\xi \text{ in } L^2(\mu), \quad \mu(\mathbb{T}) = 1, \quad b \equiv 1, \quad b_* = U_1^* b \equiv \bar{\xi}$$

- Let μ_α be the spectral measure of U_α corresponding to the vector b .
- Want to find a unitary operator $\mathcal{V}_\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha)$ such that $\mathcal{V}_\alpha b = \mathbf{1} \in L^2(\mu_\alpha)$ and such that

$$\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha.$$

Case of self-adjoint perturbations was treated earlier by Liaw–Tretter in [3].
This case is treated similarly.

Pretending to be a physicist

Let \mathcal{V}_α be an integral operator with kernel $K(z, \xi)$.

- $U_\alpha = M_\xi + (\alpha - 1)bb_*^*$, so we can rewrite the relation $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha M_\xi = M_z \mathcal{V}_\alpha - (1 - \alpha) \mathcal{V}_\alpha bb_*^*.$$

- We know that $\mathcal{V}_\alpha b = 1$, $b_* = \bar{\xi}$, so $\mathcal{V}_\alpha bb_*^*$ is an integral operator with kernel ξ

$$K(z, \xi)\xi = zK(z, \xi) - (\alpha - 1)\xi.$$

- Solving for K we get

$$K(z, \xi) = (1 - \alpha) \frac{\xi}{\xi - z} = (1 - \alpha) \frac{1}{1 - \bar{\xi}z}$$

Commutation relations and Cauchy type integrals

A general statement

Rank one commutation relations like

$$\mathcal{V}M_\xi = M_z\mathcal{V} + cb^*$$

usually give singular integral representations for \mathcal{V} .

First representation for \mathcal{V}_α

Theorem (Representation of \mathcal{V}_α)

The unitary operator $\mathcal{V}_\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha)$ such that $\mathcal{V}_\alpha b = \mathbf{1} \in L^2(\mu_\alpha)$ and such that

$$\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha.$$

is given by

$$\mathcal{V}_\alpha f(z) = f(z) + (1 - \alpha) \int_{\mathbb{T}} \frac{f(\xi) - f(z)}{1 - \bar{\xi}z} d\mu(\xi)$$

for $f \in C^1(\mathbb{T})$

Idea of the proof

- Recalling that $U_\alpha = U_1 + (\alpha - 1)bb_*^*$ rewrite $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha U_1 = M_z \mathcal{V}_\alpha + (1 - \alpha)(\mathcal{V}_\alpha b)b_*^*.$$

Idea of the proof

- Recalling that $U_\alpha = U_1 + (\alpha - 1)bb_*^*$ rewrite $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha U_1 U_1 = M_z \mathcal{V}_\alpha U_1 + (1 - \alpha)(\mathcal{V}_\alpha b)b_*^* U_1.$$

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$$\mathcal{V}_\alpha U_1 U_1 = M_z \mathcal{V}_\alpha U_1 + (1 - \alpha)(\mathcal{V}_\alpha b)b_*^* U_1.$$

- Right multiplying by U_1 and applying the above “black” identity to $\mathcal{V}_\alpha U_1$ in the right hand side, we get

$$\mathcal{V}_\alpha U_1^2 = M_z^2 \mathcal{V}_\alpha + (1 - \alpha) [(M_z \mathcal{V}_\alpha b)b_*^* + (\mathcal{V}_\alpha b)b_*^* U_1]$$

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- By induction we get

$$\mathcal{V}_\alpha U_1^n = M_z^n \mathcal{V}_\alpha + (1 - \alpha) \sum_{k=1}^n M_z^{k-1} (\mathcal{V}_\alpha b)b_*^* U_1^{n-k}.$$

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- Applying to $b \equiv 1$ and summing geometric progression we get the formula for $f(\xi) = \xi^n$, $n \geq 0$.

Idea of the proof, continued

- To get the formula for $\bar{\xi}^n$ we use $\mathcal{V}_\alpha U_\alpha^* = M_{\bar{z}} \mathcal{V}_\alpha$, which is obtained by taking adjoint in $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$.
- Extend the formula from trig. polynomials to $f \in C^1$ by standard approximation reasoning.

Idea of the proof, continued

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- Extend the formula from trig. polynomials to $f \in C^1$ by standard approximation reasoning.

A general statement

Rank one commutation relations like

$$\mathcal{V} M_\xi = M_z \mathcal{V} + cb^*$$

usually give singular integral representations for \mathcal{V} .

Singular integral operators

Recall that $\mathcal{V}_\alpha f(z) = f(z) + (1 - \alpha) \int_{\mathbb{T}} \frac{f(\xi) - f(z)}{1 - \bar{\xi}z} d\mu(\xi)$

Theorem (Regularization of the weighted Cauchy transform)

The integral operators $T_r = T_r^\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha)$ with kernels $1/(1 - r\bar{\xi}z)$, $r \in \mathbb{R}_+ \setminus \{1\}$ are uniformly bounded.

- Let $Tf(z) := \int_{\mathbb{T}} \frac{f(\xi)}{1 - \bar{\xi}z} d\mu(\xi)$; well defined for $z \notin \text{supp } f$
- Since \mathcal{V}_α is bounded, we get for $f, g \in C^1$, $\text{supp } f \cap \text{supp } g = \emptyset$

$$(Tf, g)_{L^2(\mu_\alpha)} \leq C \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu_\alpha)}$$

- By a theorem of Liaw–Treil [4] this implies uniform boundedness of the regularizations T_r if the measures μ and μ_α do not have common atoms (U_1 and U_α do not have common eigenvalues).

Singular integral operators

- Uniform boundedness of T_r together with μ_α -a.e. convergence of $T_r f$ imply existence of w.o.t.-limits $T_\pm^\alpha = \text{w.o.t.-}\lim_{r \rightarrow 1^\mp} T_r$.
- Using T_\pm^α we can rewrite the representation

$$\mathcal{V}_\alpha f(z) = f(z) + (1 - \alpha) \int_{\mathbb{T}} \frac{f(\xi) - f(z)}{1 - \bar{\xi}z} d\mu(\xi)$$

as

$$\mathcal{V}_\alpha f = [\mathbf{1} - (1 - \alpha)T_\pm^\alpha \mathbf{1}]f + (1 - \alpha)T_\pm^\alpha f.$$

- $T_\pm^\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha)$, $T_\pm^\alpha f$ is given by boundary values of $\mathcal{C}[f\mu]$, $\mathcal{C}\tau(z) = \int_{\mathbb{T}} (1 - \bar{\xi}z)^{-1} d\tau(\xi)$.
- $(\mu_\alpha)_a$ -a.e. convergence follows from classical results about jumps of Cauchy transform; $(\mu_\alpha)_s$ -a.e. convergence can be obtained from Poltoratskii's theorem about boundary values of the normalized Cauchy transform, see [10].
- For the weak convergence it is enough to have μ_α -a.e. convergence of $T_r f$ for $f \in C^1$, which can be proved using elementary methods.

Model, agreement of coordinate and parametrizing operators

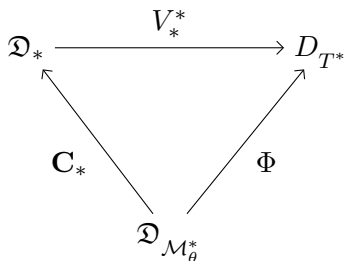
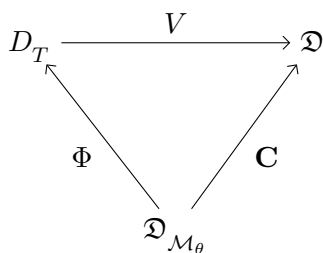
- Let T be a c.n.u. contraction, $V : \mathfrak{D}_T = \mathfrak{D}$, $V : \mathfrak{D}_{T^*} = \mathfrak{D}_*$ unitary operators (coordinate operators),
- $\theta = \theta_{T, V, V_*} \in H^\infty(\mathfrak{D} \rightarrow \mathfrak{D}_*)$ its characteristic function, $\mathcal{M}_\theta : \mathcal{K}_\theta \rightarrow \mathcal{K}_\theta$ the model operator.
- We say that unitary $\mathbf{C} : \mathfrak{D} \rightarrow \mathfrak{D}_{\mathcal{M}_\theta}$, $\mathbf{C}_* : \mathfrak{D}_* \rightarrow \mathfrak{D}_{\mathcal{M}_\theta^*}$ agree with V , V_* if

$$\mathbf{C}^* = V\Phi \Big|_{\mathfrak{D}_{\mathcal{M}_\theta}}, \quad \mathbf{C}_*^* = V_*\Phi \Big|_{\mathfrak{D}_{\mathcal{M}_\theta^*}}.$$

for a unitary $\Phi : \mathcal{K}_\theta \rightarrow \mathcal{H}$ such that $T\Phi = \Phi\mathcal{M}_\theta$

Model, agreement of coordinate and parametrizing operators

In other words, the following diagrams commute:



Model: agreement

In the Sz.-Nagy–Foiş notation

$$\mathbf{C}_* e_* = \begin{pmatrix} \mathbf{I} - \theta(z)\theta^*(0) \\ -\Delta(z)\theta^*(0) \end{pmatrix} (\mathbf{I} - \theta(0)\theta^*(0))^{-1/2} e_*, \quad e_* \in \mathfrak{D}_*,$$

$$\mathbf{C} e = \begin{pmatrix} z^{-1}(\theta(z) - \theta(0)) \\ z^{-1}\Delta(z) \end{pmatrix} (\mathbf{I} - \theta^*(0)\theta(0))^{-1/2} e, \quad e \in \mathfrak{D},$$

For the Clark case $T = T_\Gamma = T + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*U$, $V = U^*\mathbf{B}$, $V_* = \mathbf{B}$, $\mathfrak{D} = \mathfrak{D}_* = \mathbb{C}^d$ we get, noticing that $\theta(0) = -\Gamma$ that

$$\mathbf{C} e(z) = C(z)e, \quad \mathbf{C}_* e(z) = C_*(z)e,$$

where

$$C_*(z) = \begin{pmatrix} \mathbf{I} + \theta(z)\Gamma^* \\ \Delta(z)\Gamma^* \end{pmatrix} D_{\Gamma^*}^{-1},$$

$$C(z) = z^{-1} \begin{pmatrix} \theta(z) + \Gamma \\ \Delta(z) \end{pmatrix} D_\Gamma^{-1};$$

Theorem (A “universal” representation formula)

In the rank one case the adjoint Clark operator Φ^* , (C, C_* agree with Clark model) is given for $f \in C^1(\mathbb{T})$ by

$$\Phi_\gamma^* f(z) = C_*(z)f(z) + C_1(z) \int \frac{f(\xi) - f(z)}{1 - \bar{\xi}z} d\mu(\xi), \quad z \in \mathbb{T},$$

where $C_1(z) = C_*(z) - zC(z)$

Regularizing Cauchy Transform we get the following representation of the Φ^* ,

$$\Phi^* f(z) = A(z)f(z) + C_1(z)\mathcal{C}_+[f\mu](z),$$

where $A = C_* - C_1\mathcal{C}_+\mu$,

$$\mathcal{C}_\tau(z) = \int_{\mathbb{T}} \frac{1}{1 - \bar{\xi}z} d\tau(\xi).$$

\mathcal{C}_+ means boundary values of $\mathcal{C}_\tau(z)$, $z \in \mathbb{D}$.

Idea of the proof

- Write, denoting $C_2(z) := zC(z)$,

$$\begin{aligned}\mathcal{M}_{\theta_\gamma} &= M_z - C_2C^* - \theta_\gamma(0)C_*C^* \\ &= M_z + (\gamma C_* - C_2)C^*.\end{aligned}$$

Rank one perturbation of M_z ! Should get at most rank 2 commutation relation.

Idea of the proof

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$$\begin{aligned}\mathcal{M}_{\theta_\gamma} &= M_z - C_2 C^* - \theta_\gamma(0) C_* C^* \\ &= M_z + (\gamma C_* - C_2) C^*.\end{aligned}$$

Rank one perturbation of M_z ! Should get at most rank 2 commutation relation.

- Using this identity rewrite $\Phi_\gamma^* T_\gamma = \mathcal{M}_{\theta_\gamma} \Phi_\gamma^*$ as

$$\Phi_\gamma^* U + (\gamma - 1) C_* b^* U = M_z \Phi_\gamma^* + (\gamma C_* - C_2) b^* U$$

or equivalently

$$\Phi_\gamma^* U = M_z \Phi_\gamma^* + (C_* - C_2) b^* U.$$

We got rank one commutation relation!

- Commutation relations imply integral representation.

Idea of the proof, difficulties

- Formally the right side of

$$\Phi_\gamma^* U = M_z \Phi_\gamma^* + (C_* - C_2) b^* U. \quad (*)$$

acts from $L^2(\mu)$ to outside of \mathcal{K}_θ .

- To get $\Phi_\gamma^* \bar{\xi}^n$ we use the commutant relation

$$\begin{aligned} \Phi_\gamma^* U^* &= M_{\bar{z}} \Phi_\gamma^* + (C - M_{\bar{z}} C_*) b^* \\ &= M_{\bar{z}} \Phi_\gamma^* - M_{\bar{z}} (C_* - C_2) b^*, \end{aligned}$$

which cannot be obtained by taking the adjoint of (*).

- It is a miracle that the formulas for $\Phi_\gamma^* \xi^n$ and $\Phi_\gamma^* \bar{\xi}^n$ agree.

Matrix case: universal formula

Universal formula: for $b \in \text{Ran } \mathbf{B}$ and scalar $h \in C^1(\mathbb{T})$

$$(\Phi^*hb)(z) = h(z)C_*(z)\mathbf{B}^*b + C_1(z) \int_{\mathbb{T}} \frac{h(\xi) - h(z)}{1 - z\bar{\xi}} B^*(\xi)b(\xi) d\mu(\xi)$$

where, recall $C_1(z) = C_*(z) - zC(z)$

- As in the scalar case, Φ^* has Cauchy transform part, plus multiplication part.
- Cauchy transform part is easy (put $g = hb$),

$$g \mapsto C_1\mathcal{C}_+[B^*g\mu], \quad g \in \mathcal{H} \subset L^2(\mu; E).$$

where, recall

$$\mathcal{C}_{\mathcal{T}}(z) = \int_{\mathbb{T}} \frac{1}{1 - \bar{\xi}z} d\tau(\xi).$$

and \mathcal{C}_+ means boundary values of $\mathcal{C}_{\mathcal{T}}(z)$, $z \in \mathbb{D}$.

Matrix case: spectral representation with matrix weight

Consider the weighted space $L^2(B^*B\mu)$,

$$\|f\|_{L^2(B^*B\mu)}^2 := \int_{\mathbb{T}} (B(\xi)^*B(\xi)f(\xi), f(\xi))_{\mathbb{C}^d} d\mu(\xi)$$

- The operator $\mathcal{U} : L^2(B^*B\mu) \rightarrow \mathcal{H}$, $\mathcal{U}f = Bf$ is unitary.
- The adjoint Clark operator $\Phi^* : L^2(B^*B\mu) \rightarrow \mathcal{K}_\theta$ is given by

$$\Phi^* f = \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} f + \begin{pmatrix} (\mathbf{I} + \theta_\Gamma \Gamma^*) D_\Gamma^{-1} F^{-1} \\ \Delta D_\Gamma^{-1} (\Gamma^* - \mathbf{I}) \end{pmatrix} \mathcal{C}_+[B^*Bf\mu],$$

where

$$F = \mathcal{C}_+[B^*B\mu], \quad \Psi_2(z) = \Delta D_\Gamma^{-1} (\Gamma^* + (\mathbf{I} - \Gamma^*)F(z)) P_{\text{Ran } B(z)^*}$$

What remains to be done?

- Formula for the Clark operator $\Phi : \mathcal{K}_\theta \rightarrow \mathcal{H}$: it is done for the rank one case.
- For general rank one case one needs to have a matrix version of Poltoratskii's theorem about μ -a.e. convergence of the normalized Cauchy transform.
- As shown by Kapustin–Poltoratskii [2], direct generalization of the scalar result fails. However, it was shown by them in [2] that one gets the matrix result after a multiplication by a projection-valued function (but only for the case of singular measure).
- One needs to extend the results from [2] about μ -a.e. convergence to general measures.
- Infinite rank perturbations: trace class, Hilbert–Schmidt?

Comparison with Clark model

- D. Clark started with model operator K_θ , (θ inner $\iff \mu$ is purely singular) and considered it all unitary rank one perturbations.
- In our model it corresponds considering operator $U_\gamma = U_1 + (\gamma - 1)bb_1^*$, $\gamma = -\theta(0)$, then all unitary rank one perturbations are exactly the operators U_α , $|\alpha| = 1$.
- Clark measures $\tilde{\mu}_\alpha$ are the spectral measures of the operators U_α .
- If $\theta(0) = 0$ then $\tilde{\mu}_\alpha = \mu_\alpha$ and the Clark operators coincide with ours.
- If $\theta(0) \neq 0$ $\tilde{\mu}_\alpha$ is a multiple μ_α , and the operators differ by a factor $c(\gamma)$.
- In Clark model $\tilde{\mu}_\alpha$ is not a probability measure, $|c(\gamma)|$ compensate for that.

Comparison with Sarason's model

- D. Sarason in [11] presented a unitary operator between $H^2(\mu) = \overline{\text{span}}\{z^n : n \in \mathbb{Z}_+\}$ and the de Branges space $\mathcal{H}(\theta)$; like Clark, he started with a model operator in \mathcal{K}_θ
- The space $\mathcal{H}(\theta) \subset H^2$ is defined as a range $(I - T_\theta T_{\theta^*})^{1/2} H^2$ endowed with the *range norm* (the minimal norm of the preimage); $T_\varphi : H^2 \rightarrow H^2$ is a Toeplitz operator, $T_\varphi f = P_{H^2}(\varphi f)$.
- If θ is an extreme point of the unit ball in H^∞

$$\left(\int_{\mathbb{T}} \ln(1 - |\theta|^2) |dz| = -\infty \iff \int_{\mathbb{T}} \ln w |dz| = -\infty, w \text{ density of } \mu \right)$$
 then $\mathcal{H}(\theta)$ is canonically isomorphic to the model space \mathcal{K}_θ in the de Branges–Rovnyak transcription, see [9].
- His measure μ coincides with the Clark measure $\tilde{\mu}_\alpha$,

$$\alpha = \frac{1 + \gamma}{1 + \bar{\gamma}};$$

the formulas are the same as Clark's.

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