# AN ANALOG OF SOME TITCHMARSH THEOREM FOR THE FOURIER TRANSFORM ON LOCALLY COMPACT VILENKIN GROUPS

26th St.Petersburg Summer Meeting in Mathematical Analysis
June 25th, 2017

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3 июля 2017 г.

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The main themes of the talk are some analogues of one classical Titchmarsh theorem on description of the image under the Fourier transform of a class of functions satisfying the Lipschitz condition in $L^2$ and the sharp estimate (with exact constant) for decreasing of certain Fourier transforms of $L^2$ functions in mean.

Suppose that $f(x)$ is a function in the $L^2(\mathbb{R})$ space (all functions below are complex-valued), $\|\cdot\|_{L^2(\mathbb{R})}$ is the norm of $L^2(\mathbb{R})$, and $\alpha$ is an arbitrary number in the interval (0, 1).

## Definition 1

*A function $f(x)$ belongs to the Lipschitz class $Lip(\alpha, 2)$ if*

$$\|f(x-t) - f(x)\|_{L^2(\mathbb{R})} = O(t^\alpha)$$

*as $t \to 0$.*

## Theorem 1 ([T], Theorem 85)

*If $f(x) \in L^2(\mathbb{R})$ and $\widehat{f}(\lambda)$ is its Fourier transform, then the conditions*

$$f \in Lip(\alpha, 2), \qquad 0 < \alpha < 1,$$

*and*

$$\int\limits_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 \, d\lambda = O(r^{-2\alpha})$$

*as $r \to \infty$ are equivalent.*

[T] Titchmarsh E. C., *Introduction to the theory of Fourier integrals*, Oxford: Clarendon Press, 1937.

There are many analogues of Theorem 1: for the Fourier transform on noncompact Riemannian rank 1 symmetric spaces, in particular for the Fourier transform on the Lobachevskiĭ plane; for the Fourier – Jacobi transform; for the Fourier – Dunkl transform and etc. See, for example:

[Pl1] Platonov S. S., *The Fourier transform of functions satisfying the Lipschitz condition on rank 1 symmetric spaces 1*, Sib. Math. J., **46**:6 (2005), 1108–1118.
[Y] Younis M. S., *Fourier transform of Lipschitz functions on the hyperbolic plane*, Internat. J. Math.& Math. Sci., **21**:2 (1998), 397-401.
[DH] Daher R., Hamma M., *An analog of Titchmarsh's theorem of Jacobi transform*, Int. J. of Math. Anal., **6**:17-20 (2012), 975-981.
[M] Maslouhi M., *An analog of Titchmarsh's theorem for the Dunkl transform*, Integral Transforms Spec. Funct., **21**:10 (2010), 771-778.

We obtain some analogue of Theorem 1 for the Fourier transform on locally compact Vilenkin groups.

Let us present necessary definitions from harmonic analysis on locally compact Abelian groups (see, for example, [X-P], [Rud]).

[H-R] E. Hewitt and K. A. Ross, Abstract harmonic analysis, vol. I: Structure of topological groups. Integration theory, group representations, Grundlehren Math. Wiss., vol. 115, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg 1963, viii+519 pp

[Rud] Rudin W., *Fourier analysis on groups.* New York and London: Interscience Publishers, 1962.

Let $G$ be a locally compact Abelian group. A character of $G$ is a continuous complex-valued function $\chi(x)$ on $G$ such that $|\chi(x)| = 1$ and $\chi(x + y) = \chi(x)\chi(y)$ for any $x, y \in G$. Let $\Gamma$ be the set of all characters of $G$. The set $\Gamma$ equipped with the compact-open topology and the operation of point-wise multiplication of characters becomes an LCA-group which is said to be the dual group of $G$. We note that the group operation in the group $G$ is always written additively and the operation in the dual group $\Gamma$ is written multiplicatively.

## Definition 2

*A locally compact Abelian group $G$ is said to be a locally compact Vilenkin group if there exists a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n\in\mathbb{Z}}$ (that is $G_{n+1} \subsetneqq G_n$) such that $\bigcup_{n\in\mathbb{Z}} G_n = G$ and $\bigcap_{n\in\mathbb{Z}} G_n = \{0\}$.*

The factor group $G_n/G_{n+1}$ is a finite Abelian group. Let $d_n$ be the order of the group $G_n/G_{n+1}$, then $d_n \geq 2$. Examples of locally compact Vilenkin groups are the group p-adic numbers and, more generally, the additive group of a local field, see [Taibl].

[Taibl] Taibleson M. H., *Fourier analysis on local fields*, Math. Notes, vol. 15, Prinston Univ. Press, 1975.

In what follows $G$ is a locally compact Vilenkin group, $\Gamma$ its dual group. For any $n \in \mathbb{Z}$ let $\Gamma_n$ be the annihilator of $G_n$, that is

$$\Gamma_n = \{\chi \in \Gamma : \chi(x) = 1 \text{ for any } x \in G_n\}.$$

It follows from the properties of dual groups and the annihilators of subgroups (see [H-R, (23.24), (23.29)]) that $\Gamma_n$ is a compact open subgroup of $\Gamma$, the sequence of subgroups $\{\Gamma_n\}_{n\in\mathbb{Z}}$ is strictly increasing, $\bigcap_{n\in\mathbb{Z}} \Gamma_n = \{1\}$ and $\bigcup_{n\in\mathbb{Z}} \Gamma_n = \Gamma$.

We choose Haar measures $dx$ on $G$ and $d\chi$ on $\Gamma$ so that

$$\int_{G_0} dx = \int_{\Gamma_0} d\chi = 1. \tag{1}$$

We denote by $\mu(A)$ the Haar measure of a subset $A \subset G$, and by $\lambda(B)$ the Haar measure of a subset $B \subset \Gamma$.
For every $s \in \mathbb{Z}$ we define the the number $m_n$ by

$$m_n := \begin{cases} d_1 d_2 \dots d_n & \text{if } n > 0, \\ 1 & \text{if } n = 0, \\ d_0^{-1} d_{-1}^{-1} \dots d_{-n+1}^{-1} & \text{if } n < 0. \end{cases} \tag{2}$$

Then

$$\mu(G_n) = \frac{1}{m_n}, \quad \lambda(\Gamma_n) = m_n.$$

Let $L^p(G)$, $1 \le p < \infty$, be a Banach space of all measurable $\mathbb{C}$-valued functions $f(x)$ on $G$ with finite norm

$$\|f\|_p = \|f\|_{L^p(G)} := \left( \int_G |f(x)|^p \, dx \right)^{1/p}.$$

Similarly, let $L^p(\Gamma)$ be a Banach space of all measurable $\mathbb{C}$-valued functions $g(\chi)$ on $\Gamma$ with finite norm

$$\|g\|_p = \|g\|_{L^p(\Gamma)} := \left( \int_\Gamma |g(\chi)|^p \, d\chi \right)^{1/p}.$$

As usual, functions from the spaces $L^p$ are considered up to their values on a set of measure 0.

For any function $f(x) \in L^1(G)$, by the Fourier transform of $f$ we mean the function $\widehat{f}(\xi)$ on $\Gamma$ defined by the formula

$$\widehat{f}(\chi) := \int_G f(x)\,\chi(x)\,dx, \qquad \chi \in \Gamma.$$

If $f \in L^2(G)$, then its Fourier transform $\widehat{f}(\xi)$ can be defined as the limit in $L^2(G)$ of a sequence of the functions

$$\widehat{f_n}(\chi) := \int_{G_n} f(x)\,\chi(x)\,dx$$

as $n \to \infty$. The Fourier transform $F : f(x) \mapsto \widehat{f}(\chi)$ is a linear isomorphism of the space $L^2(G)$ into the space $L^2(\Gamma)$, and for any function $f \in L^2(G)$ we have the Parseval's identity

$$\|F(f)\|_{L^2(\Gamma)} = \|f\|_{L^2(G)}.$$

For a function $f(x)$ on $G$ and for any $h \in G$ let

$$(\tau_h f)(x) := f(x - h).$$

The operator $\tau_h$ is called the translation operator. If $f \in L^2(G)$ and $F(f)(\chi) = \widehat{f}(\chi)$ is its Fourier transform, then we have:

$$F(\tau_h f)(\chi) = \chi(h)\,\widehat{f}(\chi). \qquad (3)$$

For $f \in L^2(G)$ and $n \in \mathbb{N}$ let

$$\omega_2(f;n) := \sup\{\|f - \tau_h f\|_2 : h \in G_n\}.$$

The sequence of numbers $\{\omega_2(f;n)\}_{n\in\mathbb{N}}$ is called the modulus continuity of $f$ in the space $L^2(G)$.
Let be $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ a sequence of real numbers monotonously decreasing to zero (that is (i) $\omega_n \geq 0$; (ii) $\omega_n \geq \omega_{n+1} \quad \forall n \in \mathbb{N}$; (iii) $\omega_n \to 0$ as $n \to \infty$).

**Definition 3**

*A function $f(x)$ belongs to the space $H_2^\omega(G)$, if $f \in L^2(G)$ and for some constant $c = c(f) > 0$ we have*

$$\omega_2(f;n) \leq c\,\omega_n, \qquad n \in \mathbb{N}.$$

Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ and $\omega' = \{\omega'_n\}_{n\in\mathbb{N}}$ be sequences of real numbers monotonously decreasing to zero. The sequences $\omega$ and $\omega'$ will be called equivalent if we have

$$c_1\,\omega_n \le \omega'_n \le c_2\,\omega_n, \qquad n\in\mathbb{N}$$

for some positive constants $c_1$ and $c_2$. It can be proved that every space $H_2^{\omega}(G)$ is nonzero, and $H_2^{\omega}(G_p) = H_2^{\omega'}(G)$ if and only if the sequences $\omega$ and $\omega'$ are equivalent.

The main results of the talk are the next theorems.

## Theorem 2

*For every $f \in L^2(G)$ we have the inequality*

$$\left( \int\limits_{\Gamma \setminus \Gamma_n} |\widehat{f}(\chi)|^2 \, d\chi \right)^{1/2} \leq \frac{1}{\sqrt{2}} \omega_2(f; n), \qquad n \in \mathbb{N}, \tag{4}$$

*where constant $\frac{1}{\sqrt{2}}$ in (4) is exact.*

The following theorem is an analogue of the Tichmarsh theorem.

## Theorem 3

*Let* $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ *be any sequence of real numbers monotonously decreasing to zero. Then the next conditions are equivalent:*

$$f \in H_2^{\omega}(G) \tag{5}$$

*and*

$$\left( \int\limits_{\Gamma\setminus\Gamma_n} |\widehat{f}(\chi)|^2 \, d\chi \right)^{1/2} \leq c\,\omega_n, \qquad n \in \mathbb{N}, \tag{6}$$

*where* $c = c(f)$ *is some positive constant.*

For the special case when $G = \mathbb{Q}_p$ is the group of $p$-adic numbers, the Theorems 2 and 3 was proved in
[Pl2] Platonov S. S., *An analogue of the Titchmarsh theorem for the Fourier transform on the group of p-adic numbers*, p-Adic Numbers, Ultrametric Analysis and Appl., **9**:2 (2017), 158–164.

### Lemma 1

*Let $\chi$ be a character of группы $G$, $n \in \mathbb{Z}$. Then*

$$\int_{G_n} \chi(x)\,dx = \begin{cases} \mu(G_n), & \text{if } \chi \in \Gamma_n, \\ 0, & \text{if } \chi \notin \Gamma_n. \end{cases}$$

**Proof of Theorem 2**

1) Let $f \in L^2(G)$, $h \in G_n$, $n \in \mathbb{N}$. By definition of the modulus of continuity we have

$$\omega_2(f;n) := \sup\{\|f - \tau_h f\|_2 : h \in G_n\}. \tag{7}$$

It follows from (3) that

$$F(f - \tau_h f)(\xi) = (1 - \chi_p(\xi h))\,\widehat{f}(\xi), \tag{8}$$

then, using the Parseval's identity, we have

$$\|f - \tau_h f\|_2^2 = \int\limits_{\Gamma} |1 - \chi(h)|^2\,|\widehat{f}(\chi)|^2\,d\chi. \tag{9}$$

If $\chi \in \Gamma_n$, $h \in G_n$, then $\chi(h) = 1$. Hence, the equality (9) can be rewritten in the form

$$\|f - \tau_h f\|_2^2 = \int\limits_{\Gamma \setminus \Gamma_n} |1 - \chi(h)|^2 \, |\widehat{f}(\chi)|^2 \, d\chi. \qquad (10)$$

Integrating the equality (10) with respect to $h \in G_n$, we obtain

$$\int\limits_{G_n} \|f - \tau_h f\|_2^2 \, dh = \int\limits_{\Gamma \setminus \Gamma_n} \left( \int\limits_{G_n} |1 - \chi(h)|^2 \, dh \right) |\widehat{f}(\chi)|^2 \, d\chi. \qquad (11)$$

It follows from $|\chi(h)| = 1$ that

$$|1 - \chi(h)|^2 = 2 - 2\,\mathrm{Re}\,\chi(h). \tag{12}$$

It follows from Lemma 1 that

$$\int_{G_n} \chi(h)\,dh = 0 \quad \text{for } \chi \in \Gamma \setminus \Gamma_n, \tag{13}$$

hence it follows from (12) and (13) that

$$\int_{G_n} |1 - \chi(h)|^2\,dh = \int_{G_n} (2 - \mathrm{Re}\,\chi(h))\,dh = 2\int_{G_n} dh = 2\mu(G_n). \tag{14}$$

From (11) and (14) it follows that

$$\int_{G_n} \|f - \tau_h f\|_2^2\,dh = 2\mu(G_n) \int_{\Gamma\setminus\Gamma_n} |\widehat{f}(\chi)|^2\,d\chi. \tag{15}$$

On the other hand, since $\|f-\tau_h f\|_2 \le \omega_2(f;n)$ for $h \in G_n$, then

$$\int\limits_{G_n} \|f-\tau_h f\|_2^2 \, dh \le (\omega_2(f;n))^2 \int\limits_{G_n} dh = \mu(G_n)\, (\omega_2(f;n))^2 . \quad (16)$$

It follows from (15) and (16) that

$$2\mu(G_n) \int\limits_{\Gamma\setminus\Gamma_n} |\widehat{f}(\chi)|^2 \, d\chi \le \mu(G_n)\, (\omega_2(f;n))^2 ,$$

which implies that the inequality (4) holds.

2) We claim that the constant $\frac{1}{\sqrt{2}}$ in (4) is exact.
For any $n \in \mathbb{Z}$ and $a \in G$ let
$G_n(a) := a + G_n = \{x \in G : x - a \in G_n\}$. In particular, $G_n(0) = G_n$. For every $s \in \mathbb{N}$ let $\varphi_s$ be the characteristic function of the subset $G_s$. Then $\|\varphi_s\|_2^2 = \mu(G_s)$.

It can be proved that

$$\int\limits_{\Gamma\setminus\Gamma_n} |\widehat{\varphi}_s(\chi)|^2 \, d\chi = \begin{cases} \frac{1}{2}\left(1 - \frac{m_n}{m_s}\right)(\omega_2(\varphi_s; n))^2 & \text{for } n < s, \\ 0 & \text{for } n \geq s, \end{cases} \qquad (17)$$

where $m_n$ and $m_s$ is defined in (2). Since $\frac{m_n}{m_s} \leq 2^{n-s}$, then it follows from (17) that, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, for sufficiently large $s$ we have the inequality

$$\left( \int\limits_{\Gamma\setminus\Gamma_n} |\widehat{\varphi}_s(\chi)|^2 \, d\chi \right)^{1/2} \geq \frac{1}{\sqrt{2}}(1 - \varepsilon)\, \omega_2(\varphi_s; n),$$

which implies that the constant $\frac{1}{\sqrt{2}}$ in (4) is exact.

Using the results of the paper

[Rub] Rubinshtein A. I., *Moduli of continuity of functions, defined on a zero-dimensional group*, Math. Notes., **23** (1978), 205-211.

we can prove the next

## Proposition 1

*Let $\{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers monotonously decreasing to zero. Then there exist a function $f \in L^2(G)$ such that*

$$\omega_2(f;n) = \omega_n \qquad \forall n \in \mathbb{N}. \tag{18}$$

### Corollary 1

*For any sequence $\{\omega_n\}_{n\in\mathbb{N}}$ of real numbers monotonously decreasing to zero the space $H_2^\omega(G)$ is nonzero.*

### Proposition 2

*Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ and $\omega' = \{\omega'_n\}_{n\in\mathbb{N}}$ be sequences of real numbers monotonously decreasing to zero. Then $H_2^\omega(G) = H_2^{\omega'}(G)$ if and only if the sequences $\omega$ and $\omega'$ are equivalent.*

## Proof of Theorem 3

It follows from Theorem 2 that (5) entails (6).
Let $f \in L^2(G)$ and we assume that (6) holds. Arguing as in the proof of Theorem 2, we obtain that for any $h \in G_n$ the equality (11) holds. It follows from (11), using the inequalities $|1 - \chi(h)| \leq 2$ and (6), that

$$\|f - \tau_h f\|_2^2 = \int\limits_{\Gamma \setminus \Gamma_n} |1 - \chi(h)|^2 |\widehat{f}(\chi)|^2 \, d\chi \leq 4 \int\limits_{\Gamma \setminus \Gamma_n} |\widehat{f}(\chi)|^2 \, d\chi \leq 4c^2 \omega_n^2 \tag{19}$$

for $h \in G_n$, $n \in \mathbb{N}$. Taking in (19) the infinum over all $h \in G_n$, we obtain that

$$\omega_2(f; n) \leq 2c\, \omega_n, \qquad n \in \mathbb{N},$$

that is the condition (5) holds.
Hence the conditions (5) and (6) are equivalent.