An L_1-estimate for Calderon commutators, and some applications

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Vitushkin approximation theorem. Conjecture of Verdera (1980)

Vitushkin A.G. The analytic capacity of sets in problems of approximation theory. Russian Math. Surveys. 22:6 (1967).

X is a compact subset of $\mathbb{C}$; $f \in C(\mathbb{C}) \wedge \bar{\partial} f = 0$ on X^o, Q_δ is a square, γ is the analytic capacity.

$$f \in R(X) \Longleftrightarrow \forall Q_\delta : \left| \int_{Q_\delta} f(z) dz \right| \leqslant A\omega(f, \delta)\gamma(Q_\delta \setminus X),$$

Bianalytic functions: $\bar{\partial}^2 f = 0$ on open subsets of $\mathbb{C}$.

Theorem 1. (*Mazalov M. Ya. Sbornic Math. 195:5 (2004).*)

$\forall X \forall f \in C(\mathbb{C}) \wedge \bar{\partial}^2 f = 0$ on $X^o \Rightarrow f \in R^2(X)$ (by $F_1 \bar{z} + F_2$: pol. off X).

. **Analytic and bianalytic capacities**

$$\gamma(K) \overset{def}{=} \sup_F \Big\{ |\langle \overline{\partial} F, 1\rangle| : \mathrm{Spt}(\overline{\partial} F) \subset K,\ \|F\|_{\mathrm{L}_\infty} \leqslant 1,\ F(\infty) = 0 \Big\}.$$

Properties: $\gamma(B_\delta) = \delta$; semiadditivity: $\gamma(K_1 \bigcup K_2) \leqslant A(\gamma(K_1) + \gamma(K_1))$.
Tolsa. X. Acta Math., 190:1 (2003).

Bianalytic functions: fundamental solution $\pi^{-1}\overline{z}/z$ (bounded).

Analysis of Th.1 proof: natur. appear capacities such as ($z = x + iy$)

$$\gamma_{2,y}(K) \overset{def}{=} \sup_F \Big\{ |\langle \overline{\partial}^2 F, y\rangle| : \mathrm{Spt}(\overline{\partial}^2 F) \subset K,\ \|F\|_{\mathrm{L}_\infty} \leqslant 1 \Big\}.$$

Properties: $\gamma_{2,y}(B(0,\delta)) \sim \delta$; in general $\gamma_{2,y}$ is not semiadditive!

$K = \{z_1, \ldots, z_n\}$: $F(z) = \sum_{j=1}^{n} \lambda_j \frac{\overline{z - z_j}}{z - z_j}$, $\|F\|_{\mathrm{L}_\infty} \leqslant 1$, $\sum_{j=1}^{n} |\lambda_j y_j| \to \max$.

The basic special case:

K is a finite subset of a Lipschitz graph with a small constant.

Theorem 2. For $K \subset \Gamma$ where Γ: $y = g(x)$, $\|g'\|_{L_\infty} \leqslant 1$ we have $\sup_{K\subset\Gamma} \gamma_{2,y}(K) \geqslant A\text{Var}(\Gamma)$.

Using a dual extremal problem for singular integrals, we reduce Theorem 2 to an L_1-estimate for Calderon commutators:

$$F(z) = v.p. \int_\Gamma \frac{\psi(\tau) - \psi(x)}{(t-z)^2} f(t)dt$$

where $z = x + iy$ and $t = \tau + i\nu \in \mathbb{C}$. We have (on Γ):

$$\|F\|_{L_1} \leqslant A\|f\|_{L_2}\|\psi'\|_{L_2}. \tag{1}$$

An idea for a proof of (1). Consider g' and ψ' be comp. supp. We apply the identity from:

Calderon A.P. Cauchy integrals on Lipschitz curves and related operators. Proc. Mat. Acad. USA. 1977, v. 74, n. 1.

It is sufficient to estimate the integral ($|\eta| \equiv 1$):

$$\int_{-\infty}^{+\infty} \left(\lim_{\varepsilon \downarrow 0} \int_{\Gamma} \frac{(\psi(x) - \psi(\tau)) f(x) \eta(\tau)}{(t - z + i\varepsilon)^2} dz_x \right) d\tau =$$

$$\int_{-\infty}^{+\infty} \psi'(u) du \left\{ -f_+(z_u) \eta_-(z_u) + \int_{\Gamma_u} [f'_+(t) \eta_+(t) + f_-(t) \eta'_-(t)] dt_\tau \right\},$$

where $z_u = u + ig(u)$, Γ_u:$\{y = g(x),\ x > u\}$, $f = f_+ + f_-$, $\eta = \eta_+ + \eta_-$; f_+, η_+, and f_-, η_- are boundary values of Cauchy integrals above and below Γ, respectively.

From Cauchy-Bunyakowsky, integr. by parts, and L_2-boundness of the Cauchy integral, it is suffic. to obtain $\|R\|_{L_2} \leqslant A_1 \|f\|_{L_2}$ where

$$R(u) = \int_{\Gamma_u} f_+(t)\eta'_+(t)dt_\tau.$$

Using Lusin integral (Littlewood-Paley theory), we obtain:

$$\|R\|^2_{L_2} \leqslant A_2 \int\int_{D^+} |f_+(z)|^2 |\eta'_+(z)|^2 dist(z, \Gamma) dxdy \qquad (2)$$

where D^+ is the set of points above Γ.

But $\|\eta_+\|_{BMO} \leqslant A$, so $|\eta'_+(z)|^2 dist(z, \Gamma)$ is a Carleson measure (with resp. to Γ), and by the Carleson embedding theorem, the right-hand member of (2) is no more than $A_3 \|f\|^2_{L_2}$, and we obtain (1).

It may be interesting to consider generalizations of (1) (incl. R^d).