Resonances for Stark operators

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Abstract. We consider 1Dim Stark operators perturbed by compactly supported potentials. We discuss global properties of resonances for these operators.

Consider the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$, $d \geqslant 1$ where the potential V is real compactly supported. Define the resolvent $R(\lambda) = (H - \lambda)^{-1}$, which meromorphic in the cut domain $\Lambda_1 = \mathbb{C} \setminus [0, \infty)$ with a finite number of poles on $(-\infty, 0]$. Each pole is eigenvalue. We introduce the two-sheeted Riemann surface Λ of $\sqrt{\lambda}$ obtained by joining the upper and lower rims of two copies of the cut plane $\mathbb{C} \setminus [0, \infty)$ in the usual (crosswise) way. Consider the function $F(\lambda) = (R(\lambda)f, g)$ on the first sheet Λ_1, where $f, g \in L^2(\mathbb{R}^d)$ are compactly supported. The function F has a meromorphic extension from the first sheet Λ_1 on the second sheet Λ_2 of the Riemann surface $\sqrt{\lambda}$. Each pole on the second sheet Λ_2 is a resonance for H.

Resonances for the multidimensional case ($d \geqslant 2$) were studied by Melrose, Sjöstrand, and Zworski and other. We describe the main results for odd d:

$$Cr^{\frac{d}{2}} \leqslant \mathcal{N}(r) \leqslant C_* r^{\frac{d}{2}} \qquad \forall\ r \geqslant 1,$$

for some constants C, C_* depending from V, d only and $\mathcal{N}(r)$ is the number of poles in $\mathbb{C}$ (resonances of H) having modulus $\leqslant r$ and counted according to multiplicity.

We discuss the 1D case. Consider the Schrödinger operator $H = H_0 + q$ on $\mathbb{R}$, where $H_0 y = -y''$ and and $V \in L^1(\mathbb{R})$ is compactly supported. We define the operator $Y_0(\lambda) = |V|^{\frac{1}{2}} R_0(\lambda) V^{\frac{1}{2}}$, where

$$V = |V|^{\frac{1}{2}} V^{\frac{1}{2}}, \quad R_0(\lambda) = (H_0 - k^2)^{-1}, \qquad k \in \mathbb{C}_+.$$

Each $Y_0(k), k \in \mathbb{C}_+$, belongs to the trace class and thus we can define the determinant:

$$D(k) = \det(I + Y_0(k)), \qquad k \in \mathbb{C}_+. \tag{1}$$

The function D is analytic in $\mathbb{C}_+$ and has asymptotics $D(k) = 1 + o(1)$ as $|k| \to \infty$ uniformly in $\arg k \in [0, \pi]$. The function D has a finite number of zeros in $\mathbb{C}_+$. Each zero k_o of D in $\mathbb{C}_+$ belongs to $i\mathbb{R}_+$ and $k_o^2 = E_o$ is an eigenvalue of H. The function $kD(k)$ is entire and by the definition a zero in $\mathbb{C}_-$ is a resonance.

It is important that $kD(k)$ of exponential type and belongs to the Cartwright class. There are a lot of results about such functions. In particular, the Levinson Theorem determines the asymptotics of the total number of zeros of D with modulus $\leqslant r$ at $r \to \infty$. Zworski [87] proved that

$$\mathcal{N}(r, D) = \frac{2Q}{\pi} r + o(r)$$

where $\mathcal{N}(r, f)$ is the total number of zeros of f with modulus $\leqslant r$ and (Q is a diameter of $\operatorname{supp} V$).

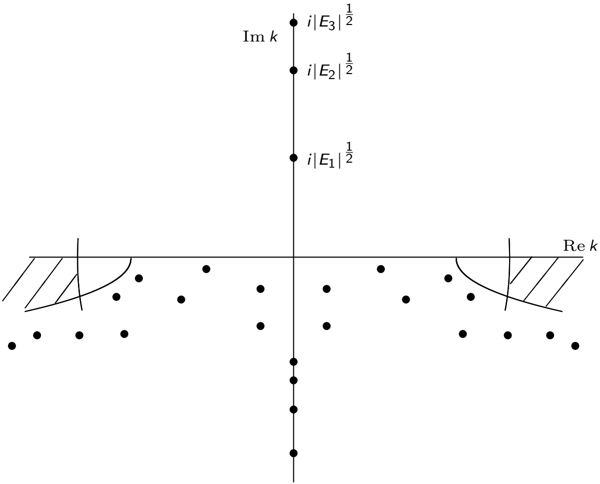

Figure: Resonances and eigenvalues $E_1 < E_2 < ...$ for the Schrödinger operator H. The resonances are marked by circles. The forbidden domain for the resonances is shaded, where there are No resonances.

A lot of papers are devoted to the resonances for the 1D Schrödinger operator: Froese [97], , Korotyaev [04], Simon [00], Zworski [87]. We recall that Zworski [87] obtained the first results about the asymptotic distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. Moreover, he considered the inverse resonance scattering for Schrödinger operator with a compactly supported potential on the real line. He solved the inverse problem to determine the potential in terms of resonances. Unfortunately, there is there is a essential mistake (the result is not correct) in Zworski's paper. Inverse problems (characterization, recovering, plus uniqueness) in terms of resonances were solved by Korotyaev for the Schrödinger operator with a compactly supported potential on the real line [05] and the half-line [04].

The perturbed Stark operators. Consider the operator $H_{\mathcal{E}} = -\Delta + (x, \mathcal{E}) + V$ on $L^2(\mathbb{R}^d), d \geqslant 1$, where $\mathcal{E} \in \mathbb{R}^d$ is an external constant electric field and the potential $V = V(x), x \in \mathbb{R}^d$ is real compactly supported.

If $\mathcal{E} = 0$, then spectrum of H_0 is a union of abs. cont. part $= \mathbb{R}_+$ and negative discrete eigenvalues.

If $\mathcal{E} \neq 0$, then spectrum of $H_{\mathcal{E}}$ is abs. cont. $= \mathbb{R}$. There are NO eigenvalues.

Scattering in an external constant electric field was studied by different authors. Avron-Herbst [77] proved the existence of the complete wave operators. Herbst [77] proved that they are unitary and there are no eigenvalues. Yajima [81] considered the S-matrix, when a constant electric field $\to 0$. Korotyaev [85] considered the scattering of 3 body systems in an external constant electric field. Herbst-Muller-Skibsted [95], Tamura [93] considered N body case. Korotyaev-Pushnitski [04] determined the series of trace formulas (only for dim=3).

The resonances for perturbed Stark operators. It is well known that atomic bound states below the continuum turn into resonances when subjected to a weak constant electric field. The resonances move continuously as a function of the field strength, ϕ, for small ϕ and converge to the original bound state as $\phi \to 0$. There are a lot of papers about it: Herbst-Simon [80-81], Graffi-Grecchi [78] etc.

We consider the Stark operator H_0 and the perturbed Stark operator H on $L^2(\mathbb{R})$ given by

$$H_0 = -\frac{d^2}{dx^2} + x, \qquad \text{and} \qquad H = H_0 + V,$$

x is an external electric potential.

The potential $V(x)$ satisfies

Condition V. *The potential V satisfies for some $\gamma > 0$:*

$$V \in L^2(\mathbb{R}), \quad \mathrm{supp}\, V \subset [0, \gamma] \quad 0 = \inf \mathrm{supp}\, V, \tag{2}$$

$$\int_0^\gamma e^{i2xk} V(x) dx = \frac{C_p}{(-i2k)^p} + \frac{O(1)}{k^\nu}, \qquad C_p \neq 0, \tag{3}$$

as $k \in \overline{\mathbb{C}}_+,\ |k| \to \infty$ uniformly in $\arg k \in [0, \pi]$, where $p \in (\frac{1}{2}, 1)$, $p < \nu$ and $(-ik)^p = e^{-ip\frac{\pi}{2}} k^p$.

Remark. 1) Here and below for $\alpha > 0, \lambda \in \overline{\mathbb{C}}_+$ we define

$$\lambda^\alpha = |\lambda|^a e^{i\alpha \arg \lambda}, \qquad \arg \lambda \in [0, \pi], \quad \log \lambda = \log |\lambda| + i \arg \lambda.$$

2) Let

$$V(x) = \frac{C_*}{x^{1-p}} + V_1(x), \qquad x \in [0, \gamma]$$

for some $p \in (\frac{1}{2}, 1)$, where V, V_1, V_1' satisfy (2). Then V satisfies (3) with $C_p = C_* \Gamma(p)$.

Next, we mention some results for one-dimensional perturbed Stark operators:

• the scattering theory was considered by Rejto-Sinha [76], Jensen [85], Liu [93];

• the inverse scattering problem are studied by Calogero-Degasperis [78], Kachalov-Kurylev [91], Kristensson [86], Lin-Qian-Zhang [89];

• there are a lot of results about the resonances, where the dilation analyticity techniques are used : Herbst [79], Jensen [89] etc. Note that compactly supported potentials are not treated in these papers.

It is known that the wave operators $W_\pm$ for the pair H_0, H given by

$$W_\pm = s - \lim e^{itH} e^{-itH_0} \qquad as \qquad t \to \pm\infty,$$

exist and are unitary. Thus the scattering operator $S = W_+^* W_-$ is unitary. The operators H_0 and S commute and thus are simultaneously diagonalizable:

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^{\oplus} \mathscr{H}_\lambda d\lambda, \qquad H_0 = \int_{\mathbb{R}}^{\oplus} \lambda I_\lambda d\lambda, \qquad S = \int_{\mathbb{R}}^{\oplus} S(\lambda) d\lambda;$$

here I_λ is the identity in the fiber space $\mathscr{H}_\lambda = \mathbb{C}$ and $S(\lambda)$ is the scattering matrix (which is a scalar function of $\lambda \in \mathbb{R}$ in our case) for the pair H_0, H.

We define the operator $Y_0(\lambda) = |V|^{\frac{1}{2}} R_0(\lambda) V^{\frac{1}{2}}$, where

$$V = |V|^{\frac{1}{2}} V^{\frac{1}{2}}, \quad R_0(\lambda) = (H_0 - \lambda)^{-1}, \qquad \lambda \in \mathbb{C}_\pm.$$

We shall show that each $Y_0(\lambda), \mathrm{Im}\, \lambda \neq 0$, belongs to the trace class and thus we can define the determinant:

$$D(\lambda) = \det(I + Y_0(\lambda)), \qquad \lambda \in \mathbb{C}_\pm. \tag{4}$$

Theorem 1. *i) The function* $D(\lambda), \lambda \in \mathbb{C}_+$ *is analytic and has an analytic extension into whole complex plane and satisfies*

$$D(\lambda) = 1 + \frac{iV_0}{8\sqrt{\lambda}} + \frac{O(1)}{\lambda} \qquad \text{as } |\lambda| \to \infty,\ \lambda \in \overline{\mathbb{C}}_+, \qquad (5)$$

where $V_0 = \int_{\mathbb{R}} V(x)dx$. *The S-matrix* $S(\lambda)$ *is continuous in* $\lambda \in \mathbb{R}$ *and satisfies*

$$S(\lambda) = \frac{\overline{D}(\lambda + i0)}{D(\lambda + i0)} = e^{-2\pi i \phi_{sc}(\lambda)}, \quad \lambda \in \mathbb{R},$$

where $\phi_{sc}(\lambda) = \frac{\arg D(\lambda+i0)}{\pi} = \frac{V_0 + O(\lambda^{-\frac{1}{2}})}{8\pi\sqrt{\lambda}}$ *and* $\phi_{sc}(-\lambda) = O(1/\lambda)$ *as* $\lambda \to \infty$.

Theorem *The function $D(\lambda)$ has the order $\frac{3}{2}$ and the type $\frac{4}{3}$, and satisfies*

$$|D(\lambda)| \leqslant C_0 e^{\frac{4}{3}|\lambda|^{\frac{3}{2}}}, \quad \lambda \in \mathbb{C}, \tag{6}$$

$$D(\lambda) = D(0) e^{\beta\lambda} \lim_{r\to+\infty} \prod_{|\lambda_n|\leqslant r} \left(1 - \frac{\lambda}{\lambda_n}\right) e^{\frac{\lambda}{\lambda_n}}, \quad \lambda \in \mathbb{C}, \tag{7}$$

for some constant C_0, uniformly on any compact subset of $\mathbb{C}$, where the constant $\beta = \frac{D'(0)}{D(0)}$, $\quad \operatorname{Im}\beta = \pi\phi'_{sc}(0)$, *and*

$$\frac{D'(\lambda)}{D(\lambda)} = \beta + \sum_{n\geqslant 1} \frac{\lambda}{\lambda_n(\lambda - \lambda_n)},$$

uniformly on any compact subset of $\mathbb{C} \setminus \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$.

By the definition, the zero $\lambda_n \in \mathbb{C}_-$ of D is a resonance. The multiplicity of the resonance is the multiplicity of the corresponding zero of D.

We discuss now inverse resonance problems. We show that all resonances determine the potential uniquely. It is a first result about inverse resonance problems for perturbed Stark operators.

Theorem (Inverse Resonance Problem.) *Let the perturbed Stark operators* $H_j = H_0 + V_j$, $j = 1, 2$ *act on* $L^2(\mathbb{R})$, *where each potential* V_j *satisfies Condition* V. *Assume that* H_1 *and* H_2 *have the same resonances. Then* $V_1 = V_2$.

Remark. In the case of Schrödinger operator with a compactly supported potential on the half-line all resonances determine the potential EK [04]. In the case of the real line all resonances do not determine the potential EK [05].

Theorem *Let $f \in C_0^\infty(R)$. Then*

$$\mathrm{Tr}\,(f(H) - f(H_0)) = -\frac{1}{\pi}\int_{\mathbb{R}} f(\lambda)\phi'_{sc}(\lambda)d\lambda, \tag{8}$$

$$\phi'_{sc}(\lambda) = \phi'_{sc}(0) + \frac{\lambda}{\pi}\,\mathrm{Im}\sum_{n\geqslant 1}\frac{1}{\lambda_n(\lambda-\lambda_n)}, \quad \forall\ \lambda \in \mathbb{R}, \tag{9}$$

$$-\,\mathrm{Tr}\,\big(R(\lambda) - R_0(\lambda)\big) = \beta + \sum_{n\geqslant 1}\frac{\lambda}{\lambda_n(\lambda-\lambda_n)}, \tag{10}$$

$$\phi_{sc}^{(m)}(0) = +\frac{(m-1)!}{\pi}\,\mathrm{Im}\sum_{n\geqslant 1}\frac{1}{\lambda_n^m} \tag{11}$$

where the series converges absolutely and uniformly on any compact set of $\mathbb{C}\setminus\{\lambda_n, n\geqslant 1\}$.

Remark. The equality (9) is a Breit-Wigner type formula for resonances.

Our main goal is to obtain global properties of the resonances of H. We describe the forbidden domain for resonances.

Theorem *The S-matrix $S(\lambda)$ satisfies*

$$S(\lambda) = 1 + o(1) \qquad \text{as} \quad \arg\lambda \in A = [\frac{2\pi}{3}, \pi],$$

$$|S(\lambda)| \to \infty, \qquad \text{as} \quad \arg\lambda \in B = [\varepsilon, \frac{2\pi}{3} - \varepsilon], \quad \varepsilon > 0,$$

as $\lambda \in \overline{\mathbb{C}}_+, |\lambda| \to \infty$. There are no resonances in the set

$$\left\{ \lambda \in \overline{\mathbb{C}}_+ : \ \arg \in A \cup B, \ |\lambda| \geqslant \varrho \right\}$$

for some $\varrho > 0$ large enough. Moreover,

$$\mathcal{N}(r) = \frac{4r^{\frac{3}{2}}}{3\pi}(1 + o(1)) \qquad \text{as} \quad r \to \infty. \tag{12}$$

Thus we see that the 1-Dim perturbed Stark operator H has much more resonances than the 1-Dim corresponding Schrödinger operator; in fact the number of resonances corresponds to 3-Dim for the Schrödinger operator.

Theorem (Asymptotics of resonances) *Let* $\rho_r = \pi(2r + \frac{p}{3})$ *for some integer* $r > 1$ *large enough. Then the function* $S(\lambda)$ *in the domain* $\{\lambda \in \mathbb{C}_+ : |\lambda| > \rho_r^{\frac{2}{3}}\}$ *have only simple zeros* $\lambda_n^\pm, n \geqslant r$ *labeled by* $|\lambda_r^\pm| < |\lambda_{r+1}^\pm| < ...$ *with asymptotics*

$$\lambda_n^\pm = \left(\frac{\pm 3\pi n}{2}\right)^{\frac{2}{3}} \left[1 \pm i \frac{(p+1)}{9\pi n}(\log|2\pi n| + O(1))\right]\Bigg) \quad \text{as} \quad n \to \infty$$

where $(1)^{\frac{2}{3}} = 1$ *and* $(-1)^{\frac{2}{3}} = e^{i\frac{2\pi}{3}}$*.*

Remarks. Note that $\operatorname{Im} \lambda_n^+ = c\frac{\log n}{n^{\frac{1}{3}}} + ...$ as $n \to \infty$. Thus the sequence of the resonances $\lambda_n^+ \in \mathbb{C}_+$ on the second sheet is more and more close to the real line. At the same time we have $\|Y(\lambda)\| \to 0$ as $|\lambda| \to \infty$ on the first sheet $\lambda \in \overline{\mathbb{C}}_-$, see (**??**). It means that perturbed resolvent has the residues at the simple resonances λ_n^+, which go zero very fast at large $n \to \infty$.

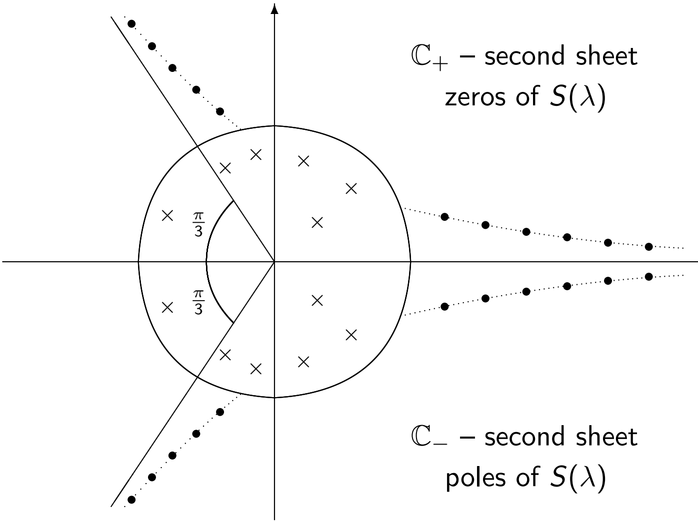

Figure:

Resonances on the non-physical sheet $\mathbb{C}_+$ (on $\mathbb{C}_-$) are zeros (poles

Finally, we remark that there are at least 3 interesting open problems connected with our results.

We list interesting open problems associated with the theory of resonances:

1) To prove the Levinson Theorem for entire functions of any order. In particular, to improve the Lindelöf theorem. Though perhaps least physical, this seems to be the deepest problem mathematically.

2) To determine the second term in the asymptotics of the Levinson Theorem.

3) To obtain sharp bounds on the number of resonances for the Stark operator in $\mathbb{R}^d$ for $d > 1$. We conjecture that this is $O(r^{1+\frac{d}{2}})$.