On a powered Bohr inequality

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(Based on the joint work with S. Ponnusamy)

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Bohr's Inequality for bounded functions

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The classical Bohr's inequality (1914) states that if a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges in $\mathbb{D}$ and $|f(z)| \le 1$ for all $z \in \mathbb{D}$, then

$$\sum_{n=0}^{\infty} |a_n| r^n \le 1 \text{ for all } r \le 1/3$$

and the constant $1/3$ cannot be improved. The constant $r_0 = 1/3$ is known as Bohr's radius. Bohr actually obtained the inequality for $r \le 1/6$, but subsequently later, Wiener, Riesz and Schur, independently established the sharp inequality for $r \le 1/3$.

H. Bohr, A theorem concerning power series, *Proc. London Math. Soc.* **13**(2) (1914), 1–5.

Conjecture of Djakov and Ramanujan (2000)

More general problem

Let $p \in (1,2)$. Find the Bohr radius r_p for

$$M_p^f(r) = \sum_{k=0}^{\infty} |a_k|^p r^k.$$

In 2000, Djakov and Ramanujan formulated the following

Conjecture

$$r_p = \inf_{a \in [0,1)} \frac{1 - a^p}{a^p(1 - a^p) + (1 - a^2)^p}.$$

P.B. Djakov, and M.S. Ramanujan, A remark on Bohr's theorem and its generalizations. *J. Anal.* **8** (2000), 65–77.

Let

$$M_p(r) = \sup_{|f|\le 1} M_p^f(r).$$

Aizenberg, Grossman and Korobeinik (2002) proved the following estimate

$$M_p(r) \le \max_{a\in[0,1]} \left[a^p + \frac{r2^p(1-a)^p}{1-r} \right]$$

A natural questions is determine the sharp value for the quantity $M_p(r)$.

L. A. Aizenberg, I. B. Grossman, Yu. F. Korobeinik, Some remarks on the Bohr radius for power series, *Izv. Vyssh. Uchebn. Zaved. Mat.*, **46** (10) (2002), 3–10; *Russian Math. (Iz. VUZ)*, **46** (10) (2002), 1–8.

Main Theorem 1.

If $0 < p \le 2$, then

$$M_p(r) = \max_{a\in[0,1]} \left[a^p + \frac{r(1-a^2)^p}{1-ra^p} \right], \quad 0 \le r \le 2^{p/2-1}, \tag{1}$$

and

$$M_p(r) < \left(\frac{1}{1-r^{2/(2-p)}} \right)^{1-p/2}, \quad 2^{p/2-1} < r < 1. \tag{2}$$

Moreover, for $p \in (1,2]$ we have

$$M_p(r) - \left(\frac{1}{1-r^{2/(2-p)}} \right)^{1-p/2} \to 0, \text{ as } r \to 1. \tag{3}$$

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Proof of Main Theorem 1.

Using the Schwarz inequality

$$\left|\frac{f(z)-a_0}{1-\overline{a_0}f(z)}\right| \leq |z|.$$

we obtain the inequality

$$\sum_{k=1}^{\infty}|a_k|^2 r^k \leq r\frac{(1-|a_0|^2)^2}{1-|a_0|^2 r}$$

Let $|a_0| = a > 0$ and $r \leq 2^{p/2-1}$. At first we suppose that $a > r^{1/(2-p)}$. In this case, the powered majorant series $M_p^f(r)$ takes the following form:

$$\begin{aligned} M_p^f(r) &= a^p + \sum_{k=1}^{\infty} \rho^k |a_k|^p \left(\frac{r}{\rho}\right)^k \\ &\leq a^p + \left(\sum_{k=1}^{\infty} (\rho^{2/p})^k |a_k|^2\right)^{p/2} \left(\sum_{k=1}^{\infty} \left(\left(\frac{r}{\rho}\right)^{2/(2-p)}\right)^k\right)^{(2-p)/2} \\ &\leq a^p + r\left(\frac{(1-a^2)^2}{1-a^2\rho^{2/p}}\right)^{p/2} \left(\frac{1}{1-(r/\rho)^{2/(2-p)}}\right)^{(2-p)/2}. \end{aligned}$$

Setting $\rho = r^{p/2} a^{(p-2)p/2}$ we obtain the inequality

$$M_p^f(r) \leq a^p + r\frac{(1-a^2)^p}{1-ra^p},$$

which proves (1) in the case $a > r^{1/(2-p)}$.

In the case $a \leq r^{1/(2-p)}$, we set $\rho = 1$ and obtain

$$M_p^f(r) = \sum_{k=0}^{\infty} |a_k|^p r^k \leq a^p + r \frac{(1-a^2)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}}.$$

Let us remark that the inequality $M_p^f(r) \leq 1$ is valid in the cases $a = 0$ and $a = r^{1/(2-p)}$. This fact can be established as a limiting case of the previous case. Finally, we let $t = a^2$. We have then to maximize the expression

$$A(t) = t^{p/2} + r \frac{(1-t)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}}, \quad t \leq r^{2/(2-p)}.$$

Using differentiation we obtain the stationary point

$$t = 1 - r^{2/(2-p)}$$

which must satisfy under the restriction $t \leq r^{2/(2-p)}$. This inequality together with $r \leq 2^{p/2-1}$ imply that $r = 2^{p/2-1}$.

proof cont...

Corollary: The Djakov-Ramanujan conjecture is true.

Problem: Is it true that

$$M_1(r) - \left(\frac{1}{1-r^2}\right)^{1/2} \to 0 \text{ as } r \to 1?$$

Bombiery and Bourgain (2004) obtained the following result:

$$M_1(r) = \left(\frac{1}{1-r^2}\right)^{1/2} + O\left(\ln^{3/2+\varepsilon} \frac{1}{1-r}\right) \text{ as } r \to 1. \tag{4}$$

Remark: (3) is an easy consequence of (4)

E. Bombieri and J. Bourgain, A remark on Bohr's inequality, *IMRN International Mathematics Research Notices*, **80**(2004), 4307–4330.

Univalent case

In the case of univalent functions, from an Area inequality

$$\sum_{k=1}^{\infty} k|a_k|^2 \leq 1,$$

it follows that

$$\sum_{k=1}^{\infty} |a_k|^p r^k < \mathrm{Li}_{\frac{p}{2-p}}(r^{\frac{2}{2-p}})^{1-p/2}.$$

In particular,

$$\sum_{k=1}^{\infty} |a_k|^p < \zeta\left(\frac{p}{2-p}\right)^{1-p/2}, \quad p > 1$$

and

$$\sum_{k=1}^{\infty} |a_k| r^k < \sqrt{\log\left(\frac{1}{1-r^2}\right)}, \quad 0 < r < 1. \tag{5}$$

Example for sharpness

A spiral like example

$$f(z) = \frac{z(1-r)^{1+\cos\alpha}}{(1-rz)^{1+e^{i\alpha}}}$$

with

$$\cos\alpha = \frac{1}{|\log(1-r)|} - 1$$

shows that (5) is sharp in order sense.

Thank you for your attention!