# On the asymptotic behavior of Jacobi polynomials with varying parameters

#### R. Zarouf, joint work with O. Szehr

Univ. of Marseille/St. Petersburg State Univ.

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# Jacobi polynomials $P_n^{(\alpha,\beta)}$

•  $P_n^{(\alpha,\beta)}, \alpha > -1, \beta > -1$ : a class of orthogonal degree-n polynomials

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n(x) P_m(x) dx = \|P_n\|^2 \delta_{n,m}.$$

• 
$$P_n^{(-1/2, -1/2)} = T_n$$
: the  $n^{th}$ -Chebyshev polynomial

• 
$$P_n^{(0,0)} = L_n$$
: the  $n^{th}$ -Legendre polynomial

$$P_n^{(\alpha,\beta)}(x) = \sum_{\mu=0}^n \binom{n+\alpha}{n-\mu} \binom{n+\beta}{\mu} \left(\frac{x-1}{2}\right)^{\mu} \left(\frac{x+1}{2}\right)^{n-\mu}$$

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## The problem, its applications

Assume  $x \in (-1, 1)$ . What is the large *n*-asymptotic behavior of:

$$P_n^{(\alpha,\beta)}(x)? \qquad P_n^{(\alpha+an,\beta)}(x), \, a > -1?$$

Most recent direct applications:

- In probablity to the so-called bead process (B. Fleming-P. Forrester- E. Nordenstam, 2010)
- In quantum physics to determine the asymptotic behavior of Hadamard walks (H. Carteret- M. Ismail- B. Richmond, 2003): the case x = 0.

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## Darboux's Theorem ( $\approx$ 1900): the case a = 0

#### Theorem

Let  $\alpha, \beta > -1$  and let  $\theta \in [\varepsilon, \pi - \varepsilon]$  then the Jacobi polynomials satisfy the following asymptotic expansion

$$P_n^{(\alpha,\beta)}(\cos(\theta)) = rac{1}{\sqrt{n\pi}}k(\theta)\cos(N\theta+\gamma) + \mathscr{O}\left(n^{-3/2}
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where

$$k(\theta) = \frac{1}{\sqrt{\pi}} \left( \sin \frac{\theta}{2} \right)^{-\alpha - \frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{-\beta - \frac{1}{2}},$$
$$= n + (\alpha + \beta + 1)/2, \qquad \gamma = -(\alpha + \frac{1}{2})\pi/2, 0 < \theta < \pi.$$

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$$N = n + (\alpha + \beta + 1)/2, \qquad \gamma = -(\alpha + \frac{1}{2})\pi/2, 0 < \theta < \pi.$$

## The general case a > -1

Group 1 (L. Chen, M. Ismail, S. Izen, ...):

- Tool: repeat Darboux's approach
- How ? Compute  $\sum_{n\geq 0} P_n^{(\alpha+an,\beta)}(x)t^n$  + apply Darboux's method
- Conclusion: very technical and most of the results are inaccurate.

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- Tool: Darboux's method.
- Corrects Group 1 for  $a \in [0, \frac{2\lambda}{1-\lambda})$ .
- Conclusion: same decay in *n* of

$$\lambda^{an} P_n^{(\alpha+an,\beta)}(1-2\lambda^2)$$

but with different (right) prefactor + application to the bead process.

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The case  $a \in \left(-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda}\right)$  (1<sup>st</sup> part)

#### Theorem

Let  $\alpha$ ,  $\beta > -1$ , a > -1 and  $\lambda \in (0, 1)$ . We have the following asymptotic expansion as  $n \to \infty$ . 1) If  $a \in \left(-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda}\right)$  then  $P_n^{(\alpha+an,\beta)}(1-2\lambda^2) = \sqrt{\frac{2}{n\pi}} \frac{\lambda^{-\alpha-an}((1-\lambda^2)(a+1))^{-\frac{\beta}{2}}}{((1-\lambda^2)((a+2)\lambda+a)((a+2)\lambda-a))^{\frac{1}{4}}} \cdot \cos\left((n+1)h(\varphi_+) + (\alpha-a)\varphi_+ + (\beta-1)\psi + \frac{\pi}{4}\right)(1+\mathcal{O}(n^{-1})).$ 

The phases  $arphi_+, h(arphi_+), \psi \in [0, 2\pi]$  depend on  $a, \lambda$  .

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The case  $a \in \left\{\frac{2\lambda}{1-\lambda}, -\frac{2\lambda}{1+\lambda}\right\}$  (2<sup>nd</sup> part)

#### Theorem

2) If 
$$a = \frac{2\lambda}{1-\lambda}$$
 then  
 $P_n^{(an+\alpha,\beta)}(1-2\lambda^2) = \frac{\lambda^{-an-\alpha}(1+\lambda)^{-\beta}}{3^{2/3}\Gamma(2/3)n^{1/3}\lambda^{1/3}(1+\lambda)^{1/3}} \left(1+\mathcal{O}(n^{-1/3})\right)$   
If  $a = -\frac{2\lambda}{1+\lambda}$  then  
 $P_n^{(an+\alpha,\beta)}(1-2\lambda^2) = \frac{\Gamma(1/3)\lambda^{-\alpha-an}(1-\lambda)^{-\beta}}{3^{2/3}\pi n^{1/3}\lambda^{1/3}(1-\lambda)^{1/3}} \left(1+\mathcal{O}(n^{-1/3})\right)$   
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$$If a = -\frac{2\lambda}{1+\lambda} then$$

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$$\cdot \frac{\sqrt{3}}{2} \left(\cos((an+\alpha)\pi) - \sqrt{3}\sin((an+\alpha)\pi)\right)$$

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# The case $a \in (\frac{2\lambda}{1-\lambda}, +\infty)$ (3<sup>rd</sup> part)

#### Theorem

3) If 
$$a \in (\frac{2\lambda}{1-\lambda}, +\infty)$$
 then the quantity

$$\lambda^{an} P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$$

decays exponentially with n.

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# The case $a \in (-1, -rac{2\lambda}{1-\lambda})$ (4<sup>th</sup> and last part)

#### Theorem

4) If  $a \in (-1, -\frac{2\lambda}{1+\lambda})$  and  $an + \alpha$  is an integer then the quantity

$$\lambda^{an} P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$$

decays exponentially with n. Else (i.e if  $an + \alpha$  is not an integer)

$$\lambda^{an} P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$$

increases exponentially with n.

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# The case $a \in (-1, -\frac{2\lambda}{1-\lambda})$ (4<sup>th</sup> and last part)

#### Theorem

4) If  $a \in (-1, -\frac{2\lambda}{1+\lambda})$  and  $an + \alpha$  is an integer then the quantity  $(an + \alpha, \beta) = -2$ 

$$\lambda^{an} P_n^{(an+lpha,\beta)}(1-2\lambda^2)$$

decays exponentially with n. Else (i.e if  $an + \alpha$  is not an integer)

$$\lambda^{an} P_n^{(an+lpha,\beta)}(1-2\lambda^2)$$

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### Integral representation for Jacobi polynomials

#### Lemma

Let n be an integer, a, lpha, eta > -1,  $\lambda \in (0, 1)$ . For any  $x \in (\lambda, 1/\lambda)$ 

$$\begin{split} \lambda^{an+\alpha} &(1-\lambda^2)^{\beta} P_n^{(an+\alpha,\beta)} (1-2\lambda^2) \\ &= \frac{1}{\pi} \Re \left\{ \int_0^{\pi} z^{\alpha+1} \frac{(1-\lambda z)^{\beta}}{z-\lambda} \left( z^{a+1} \frac{1-\lambda z}{z-\lambda} \right)^n \bigg|_{z=x e^{i\varphi}} \mathrm{d}\varphi \right\} \\ &- \frac{\sin\left(\pi(\alpha+an)\right)}{\pi} \int_0^x \frac{(1+\lambda t)^{\beta} t^{\alpha}}{t+\lambda} \left( t^{(a+1)} \frac{1+\lambda t}{t+\lambda} \right)^n \mathrm{d}t. \end{split}$$

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