On the asymptotic behavior of Jacobi polynomials with varying parameters

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Jacobi polynomials $P_n^{(\alpha,\beta)}$

- $P_n^{(\alpha,\beta)}$, $\alpha > -1, \beta > -1$: a class of orthogonal degree-n polynomials

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n(x)P_m(x)\mathrm{d}x = \|P_n\|^2\,\delta_{n,m}.$$

- $P_n^{(-1/2,-1/2)} = T_n$: the $n^{th}-$Chebyshev polynomial
- $P_n^{(0,0)} = L_n$: the $n^{th}-$Legendre polynomial
- Explicit representation for $x \in \mathbb{R}$ (extension to $\alpha, \beta \in \mathbb{R}$)

$$P_n^{(\alpha,\beta)}(x) = \sum_{\mu=0}^{n}\binom{n+\alpha}{n-\mu}\binom{n+\beta}{\mu}\left(\frac{x-1}{2}\right)^{\mu}\left(\frac{x+1}{2}\right)^{n-\mu}.$$

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The problem, its applications

Assume $x \in (-1, 1)$. What is the large $n-$asymptotic behavior of:

$$P_n^{(\alpha,\beta)}(x)? \qquad P_n^{(\alpha+an,\beta)}(x),\ a > -1?$$

Most recent direct applications:

- In probablity to the so-called bead process (B. Fleming- P. Forrester- E. Nordenstam, 2010)
- In quantum physics to determine the asymptotic behavior of Hadamard walks (H. Carteret- M. Ismail- B. Richmond, 2003): the case $x = 0$.

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Darboux's Theorem (≈ 1900): the case $a = 0$

Theorem

Let $\alpha, \beta > -1$ and let $\theta \in [\varepsilon, \pi - \varepsilon]$ then the Jacobi polynomials satisfy the following asymptotic expansion

$$P_n^{(\alpha,\beta)}(\cos(\theta)) = \frac{1}{\sqrt{n\pi}} k(\theta) \cos(N\theta + \gamma) + \mathscr{O}\left(n^{-3/2}\right),$$

where

$$k(\theta) = \frac{1}{\sqrt{\pi}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}},$$

$$N = n + (\alpha+\beta+1)/2, \qquad \gamma = -(\alpha + \frac{1}{2})\pi/2, 0 < \theta < \pi.$$

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Group 1 (L. Chen, M. Ismail, S. Izen, ...):

- Tool: repeat Darboux's approach
- How ? Compute $\sum_{n\geq 0} P_n^{(\alpha+an,\beta)}(x)t^n$ + apply Darboux's method
- Conclusion: very technical and most of the results are inaccurate.

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B. Fleming- P. Forrester- E. Nordenstam (2010).

- Tool: Darboux's method.
- Corrects Group 1 for $a \in [0, \frac{2\lambda}{1-\lambda})$.
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$$\lambda^{an} P_n^{(\alpha+an,\beta)}(1-2\lambda^2)$$

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The case $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ (1^{st} part)

Theorem

Let $\alpha,\ \beta > -1$, $a > -1$ and $\lambda \in (0,1)$. We have the following asymptotic expansion as $n \to \infty$.
1) If $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ then

$$P_n^{(\alpha+an,\beta)}(1-2\lambda^2) = \sqrt{\frac{2}{n\pi}}\frac{\lambda^{-\alpha-an}((1-\lambda^2)(a+1))^{-\frac{\beta}{2}}}{((1-\lambda^2)((a+2)\lambda+a)((a+2)\lambda-a))^{\frac{1}{4}}}$$
$$\cdot\cos\left((n+1)h(\varphi_+)+(\alpha-a)\varphi_+ +(\beta-1)\psi+\frac{\pi}{4}\right)\left(1+\mathcal{O}(n^{-1})\right).$$

The phases $\varphi_+, h(\varphi_+), \psi \in [0,2\pi]$ depend on a,λ.

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The case $a \in \left\{\frac{2\lambda}{1-\lambda}, -\frac{2\lambda}{1+\lambda}\right\}$ (2^{nd} part)

Theorem

2) If $a = \frac{2\lambda}{1-\lambda}$ *then*

$$P_n^{(an+\alpha,\beta)}(1-2\lambda^2) = \frac{\lambda^{-an-\alpha}(1+\lambda)^{-\beta}}{3^{2/3}\Gamma(2/3)n^{1/3}\lambda^{1/3}(1+\lambda)^{1/3}}\left(1+\mathcal{O}(n^{-1/3})\right)$$

If $a = -\frac{2\lambda}{1+\lambda}$ *then*

$$\begin{aligned} P_n^{(an+\alpha,\beta)}(1-2\lambda^2) \quad &= \quad \frac{\Gamma(1/3)\lambda^{-\alpha-an}(1-\lambda)^{-\beta}}{3^{2/3}\pi n^{1/3}\lambda^{1/3}(1-\lambda)^{1/3}}\left(1+\mathcal{O}(n^{-1/3})\right) \\ &\cdot\frac{\sqrt{3}}{2}\left(\cos\left((an+\alpha)\pi\right)-\sqrt{3}\sin\left((an+\alpha)\pi\right)\right) \end{aligned}$$

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The case $a \in (\frac{2\lambda}{1-\lambda}, +\infty)$ (3^{rd} part)

Theorem

3) If $a \in (\frac{2\lambda}{1-\lambda}, +\infty)$ then the quantity

$$\lambda^{an} P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$$

decays exponentially with n.

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The case $a \in (-1, -\frac{2\lambda}{1-\lambda})$ (4^{th} and last part)

Theorem

4) If $a \in (-1, -\frac{2\lambda}{1+\lambda})$ and $an+\alpha$ is an integer then the quantity

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1. Introduction
   - Definition
   - Statement of the problem, motivations
   - Known results
2. Main results
   - Statements
   - Main tool

Integral representation for Jacobi polynomials

Lemma

Let n be an integer, $a, \alpha, \beta > -1$, $\lambda \in (0,1)$. For any $x \in (\lambda, 1/\lambda)$

$$\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$$
$$= \frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=xe^{i\varphi}} \mathrm{d}\varphi\right\}$$
$$-\frac{\sin(\pi(\alpha+an))}{\pi}\int_0^x \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\left(t^{(a+1)}\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t.$$