A generalization to several variables of Löwner's Theorem on operator-monotone functions

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Operator-monotone functions

Let I be an open interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is operator-monotone if $f(A) \leq f(B)$ whenever A, B are selfadjoint operators such that $A \leq B$ and the spectra of A, B are contained in I.

Examples: $f(x) = -1/x$ is operator-monotone on $(0, \infty)$ and on $(-\infty, 0)$.

$f(x) = \sqrt{x}$ is operator-monotone on $(0, \infty)$.

$f(x) = x^2$ is not operator-monotone on $(0, \infty)$.

The Pick class

Let $\Pi = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, the upper halfplane.

The Pick class $\mathcal{P}$ is the set of holomorphic functions f on Π such that $\operatorname{Im} f \geq 0$ on Π.

Some functions in $\mathcal{P}$: $\sqrt{z}$, $-1/z$, $\log z$, $\tan z$.

For any open interval $I \subset \mathbb{R}$, define the Pick class $\mathcal{P}(I)$ of I to be the set of restrictions to I of functions $f \in \mathcal{P}$ that are analytic on I.

Löwner's theorem (1934)

Let $I \subset \mathbb{R}$ be an open interval. A real-valued function on I is operator-monotone if and only if $f \in \mathcal{P}(I)$.

Local versus global

Say that a real-valued C^1 function f on a real interval I is locally operator-monotone if, whenever $S(t)$, $0 \le t < 1$, is a C^1 curve of selfadjoint matrices with spectra contained in I,

$$S'(0) \ge 0 \qquad \Rightarrow \qquad (f \circ S)'(0) \ge 0.$$

Then $f \in C^1$ is operator-monotone on I if and only if f is locally operator-monotone on I.

Sufficiency follows from

$$f(B) - f(A) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f\left((1-t)A + tB\right) \mathrm{d}t.$$

Operator-monotonicity in 2 variables

Let E be an open set in $\mathbb{R}^2$. Say that a real-valued function f on E is operator-monotone if $f(A) \le f(B)$ whenever $A = (A_1, A_2)$ and $B = (B_1, B_2)$ are *commuting* pairs of selfadjoint operators such that $A_1 \le B_1$ and $A_2 \le B_2$ and the joint spectra of A and B are contained in E.

Say that $f \in C^1(E)$ is locally operator-monotone if, whenever $S(t) = (S_1(t), S_2(t))$, $0 \le t < 1$, is a C^1 curve of commuting pairs of selfadjoint matrices with joint spectra contained in E,

$$S'(0) \ge 0 \qquad \Rightarrow \qquad (f \circ S)'(0) \text{ exists and } \ge 0.$$

Local versus global in 2 variables

If f is operator-monotone on E then f is locally operator-monotone on E (easy). Does the converse hold?

Example

$$A = \left(\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right),$$
$$B = \left(\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \right).$$

A and B are commuting pairs of selfadjoint matrices and $A \leq B$. There is *no* commuting pair of selfadjoint matrices lying strictly between A and B.

It is unclear whether locally operator-monotone functions are operator-monotone on a general convex open set.

The Pick class in d variables

Define the d-variable Pick class $\mathcal{P}^d$ to be the set of holomorphic functions F on Π^d such that $\operatorname{Im} F \geq 0$ on Π^d.

The Pick-Agler class $\mathcal{PA}^d$ is the set of functions $F \in \mathcal{P}^d$ such that $\operatorname{Im} F(T) \geq 0$ for every d-tuple T of commuting operators having strictly positive imaginary parts.

For $F \in \mathcal{PA}^d$ there exist positive analytic kernels $A^1, \ldots, A^d$ on Π^d such that, for all $z, w \in \Pi^d$,

$$F(z) - \overline{F(w)} = (z^1 - \bar{w}^1)A^1(z,w) + \cdots + (z^d - \bar{w}^d)A^d(z,w),$$

and conversely.

Cauchy transforms of positive measures

Let $I \subset \mathbb{R}$ be an interval and let μ be a positive measure on $\mathbb{R} \setminus I$.

The *Cauchy transform* of μ is the function

$$\widehat{\mu}(z) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(s)}{s - z},$$

defined for $z \notin \mathbb{R} \setminus I$.

$\widehat{\mu}$ is locally operator monotone on I.

Proof:

Let $S(t) = A + tM + o(t)$ for $0 \le t < 1$ where $M \ge 0$.

$$\begin{aligned}
\frac{1}{t}\left(\widehat{\mu}(S(t)) - \widehat{\mu}(S(0)\right) &= \frac{1}{t}\int (s - S(t))^{-1} - (s - S(0))^{-1}\mathsf{d}\mu(s) \\
&= \int (s - S(t))^{-1}\frac{S(t) - S(0)}{t}(s - S(0))^{-1}\mathsf{d}\mu(s) \\
&= \int (s - A - tM - o(t))^{-1}M(s - A)^{-1}\mathsf{d}\mu(s) \\
&\to \int (s - A)^{-1}M(s - A)^{-1}\mathsf{d}\mu(s).
\end{aligned}$$

Thus $(\widehat{\mu} \circ S)'(0) \ge 0$. $\square$

A class of operator-monotone functions

Let E be an open rectangle in $\mathbb{R}^d$.

Let $\mathcal{M}$ be a Hilbert space and let $P = (P^1, \ldots, P^d)$ be a tuple of orthogonal projections on $\mathcal{M}$ summing to $\mathbf{1}_{\mathcal{M}}$. For $z \in \mathbb{C}^d$ let z_P denote $z^1P^1 + \cdots + z^dP^d$.

Let X be a densely defined self-adjoint operator on $\mathcal{M}$ such that $X - z_P$ is invertible for $z \in E$ and let $v \in \mathcal{M}$. The function

$$F(z) = \left\langle (X - z_P)^{-1} v, v \right\rangle_{\mathcal{M}} \qquad \text{for } z \in E$$

is operator-monotone on E.

F is a d-variable analogue of the Cauchy transform of a measure with support off E.

A 2-variable Löwner theorem

Let f be a real rational function of 2 variables and let E be an open rectangle in $\mathbb{R}^2$ on which the denominator of f does not vanish. Then f is operator-monotone on E if and only if $f \in \mathcal{P}^2$.

The proof consists in showing that f can be approximated by functions of the form $F(z) = \left\langle (X - z_P)^{-1} v, v \right\rangle_{\mathcal{M}}$ by means of a 2-variable Nevanlinna representation formula.

Our proof does not extend to dimension $d = 3$.

A d-variable Nevanlinna representation

Let $z_0 \in \Pi^d$ and let $F \in \mathcal{PA}^d$. For all but countably many automorphisms α of Π there exist a Hilbert space $\mathcal{M}$, a partition $P = (P^1, \ldots, P^d)$ of $\mathcal{M}$, a selfadjoint operator X on $\mathcal{M}$, a vector $v \in \mathcal{M}$ and a real number c such that

$$\alpha \circ F \circ \alpha(z) = c + \langle z_P v, v \rangle + \left\langle (z - z_0)_P^* (X - z_P)^{-1} (z - z_0)_P v, v \right\rangle$$

for all $z \in \Pi^d$.

If v is in the domain of X then there is a simpler representation, of the form

$$\alpha \circ F \circ \alpha(z) = c + \left\langle (X - z_P)^{-1} v, v \right\rangle.$$

The Löwner class in d variables

Let E be an open set in $\mathbb{R}^d$ and let $n \geq 1$. The Löwner class $\mathcal{L}_n^d(E)$ of E comprises all real-valued C^1 functions f on E such that, for every finite set $\{x_1, \ldots, x_n\}$ of distinct points in E, there exist positive $n \times n$ matrices $A^1, \ldots, A^d$ such that

$$A_{ii}^r = \left.\frac{\partial f}{\partial x^r}\right|_{x_i} \qquad \text{for } 1 \leq i \leq n \text{ and } 1 \leq r \leq d,$$

and

$$f(x_j) - f(x_i) = \sum_{r=1}^{d} (x_j^r - x_i^r) A_{ij}^r \qquad \text{for } 1 \leq i, j \leq n.$$

The Löwner class $\mathcal{L}^d(E)$ of E is defined to be the intersection of $\mathcal{L}_n^d(E)$ over all $n \geq 1$.

Functions in $\mathcal{L}^d(E)$ are locally operator-monotone

Consider a commuting pair $S = (S^1, S^2)$ of selfadjoint $n \times n$ matrices such that $\sigma(S) \subset E$ and $\sigma(S)$ consists of simple joint eigenvalues $x_1, \ldots, x_n$. Let $S(t)$, $0 \leq t < 1$, be a C^1 curve of commuting pairs of selfadjoint matrices such that $S(0) = S$, $\sigma(S(t)) \subset E$ for all t and

$$\Delta \stackrel{\text{def}}{=} S'(0) \geq 0.$$

If f satisfies $f(x_j) - f(x_i) = \sum_{r=1}^{2} (x_j^r - x_i^r) A_{ij}^r$ for all i, j as in the definition of $\mathcal{L}_n^2(E)$, then a calculation shows that

$$(f \circ S)'(0) = \left[\Delta_{ij}^1 A^1(i,j) + \Delta_{ij}^2 A^2(i,j) \right] \geq 0.$$

Hence f is locally operator-monotone.

Locally operator-monotone functions are in $\mathcal{L}^d(E)$

Proof is by a separation argument.

Let E be open in $\mathbb{R}^2$ and let $f \in C^1(E)$ be locally operator-monotone on E. Fix $n \geq 1$ and distinct points $x_1, \ldots, x_n \in E$. Let $\mathcal{G}$ be the set of real $n \times n$ skew-symmetric matrices Γ such that there exists a pair (A^1, A^2) of real positive $n \times n$ matrices that satisfy

$$A^r(i,i) = \frac{\partial f}{\partial x^r}(x_i),$$
$$\Gamma_{ij} = (x_j^1 - x_i^1)A^1(i,j) + (x_j^2 - x_i^2)A^2(i,j)$$

for all relevant r, i, j.

We claim that $\Lambda \stackrel{\text{def}}{=} \left[f(x_i) - f(x_j) \right]$ is in $\mathcal{G}$.

Proof that $f \in \mathcal{L}^d(E)$ continued

$\mathcal{G}$ is a nonempty closed convex set. Suppose that $\Lambda \notin \mathcal{G}$. By the Hahn-Banach theorem there is a real skew-symmetric matrix K and a $\delta \geq 0$ such that $\mathrm{tr}(\Gamma K) \geq -\delta$ for all $\Gamma \in K$ but $\mathrm{tr}(\Lambda K) < -\delta$.

Choose a curve $S(t)$, $0 \leq t < 1$, and apply the hypothesis of local operator-monotonicity. Construct $S(t)$ so that $S^r(0) = \mathrm{diag}\{x_1^r, \ldots, x_n^r\}$ and

$$(S^r)'(0)_{ij} = (x_j^r - x_i^r)K_{ji} \qquad \text{for } i \neq j.$$

Choose the diagonal entries of $(S^r)'(0)$ in a minimal way to ensure that $S'(0) \geq 0$.

Deduce a contradiction to the assumption $(f \circ S)'(0) \geq 0$ with the aid of R. J. Duffin's strong duality theorem for linear programmes. Conclude that $f \in \mathcal{L}^d(E)$.

The Löwner and Pick-Agler classes

$\mathcal{P}^d$ and $\mathcal{PA}^d$ are the Cayley transforms of the Schur and Schur-Agler classes in d variables respectively.

We have $\mathcal{PA}^2 = \mathcal{P}^2$ (Agler) and $\mathcal{PA}^d \subsetneq \mathcal{P}^d$ for $d \geq 3$ (Varopoulos).

Let E be an open set in $\mathbb{R}^d$.

Denote by $\mathcal{PA}^d(E)$ the set of functions in $\mathcal{PA}^d$ which extend analytically across E and are real on E.

Every function $f \in \mathcal{L}^d(E)$ extends to a function $F \in \mathcal{PA}^d(E)$.

A local Löwner theorem

A C^1 function f on an open set $E \subset \mathbb{R}^d$ is locally operator-monotone on E if and only if f extends to a function in $\mathcal{PA}^d(E)$.

Some questions

Are locally operator-monotone functions on a connected open set operator-monotone?

Are rational functions in d variables belonging to $\mathcal{PA}^d(E)$ operator monotone on E when E is an open rectangle in $\mathbb{R}^d$?

Reference

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