LOGARITHMIC INTERPOLATION-EMBEDDING INEQUALITY ON IRREGULAR DOMAINS

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25th Summer Meeting in Mathematical Analysis
dedicated to the memory of Victor Havin
June 26, 2016, St. Petersburg

This talk is based on the joint paper by

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Brezis-Gallouet-Wainger type inequality for irregular domains

Complex Variables and Elliptic Equations, **56**, 991–1002.

Prehistory

In 1980 Brezis and Gallouet established the unique solvability of the initial-boundary value problem for the nonlinear Schrödinger evolution equations with zero Dirichlet data on the smooth boundary of a bounded domain in $\mathbb{R}^2$ or its complementary domain. They used crucially the inequality

$$\|u\|_{L^\infty(\Omega)} \leq C\left(1 + \left(\log(1 + \|u\|_{W^{2,2}(\Omega)})\right)^{1/2}\right) \qquad (1)$$

for every $u \in W^{2,2}(\Omega)$ with $\|u\|_{W^{1,2}(\Omega)} = 1$.

Applications of this inequality to the Euler equation can be found in Chapter 13 of M.E. Taylor's book on PDEs.

The same year Brezis and Wainger extended (1) to Sobolev spaces of higher order on $\mathbb{R}^n$ in the form

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\Big(1 + \big(\log(1 + \|u\|_{W^{l,q}(\mathbb{R}^n)})\big)^{(n-k)/n}\Big) \qquad (2)$$

for every function u in $W^{l,q}(\mathbb{R}^n)$ normalized by

$$\|u\|_{W^{k,n/k}(\mathbb{R}^n)} = 1,$$

where k and l are integers, $1 \leq k < l$, $ql > n$, and $k \leq n$.

Preliminaries

Let Ω be an open domain in $\mathbb{R}^n$ such that $m_n(\Omega) < \infty$, where m_n denotes the n-dimensional Lebesgue measure on Ω. Given $p \in [1, \infty)$ and sets $E \subset G \subset \Omega$, the capacity $C_p(E, G)$ of the condenser (E, G) is defined as

$$C_p(E, G) = \inf\left\{ \int_\Omega |\nabla u|^p dx : \ u \in L^{1,p}(\Omega), \ u > 1 \text{ on } E \text{ and } u \leq 0 \text{ on } \Omega \setminus G \text{ (up to a set of } p\text{-capacity zero)} \right\}.$$

Here $L^{1,p}(\Omega)$ denotes the Sobolev space

$$\{u \in L^p(\Omega, loc) : \ |\nabla u| \in L^p(\Omega)\},$$

with the seminorm

$$\|u\|_{L^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.$$

Two isocapacitary functions

The first isocapacitary function

$$\nu_p : [0, m_n(\Omega)/2] \to [0, \infty]$$

is defined by

$$\nu_p(s) = \inf\{C_p(E,G) : E \subset G \subset \Omega, \text{ such that } m_n(E) \geq s,\ m_n(G) \leq m_n(\Omega)/2\}.$$

Clearly, ν_p is non-decreasing and the isocapacitary inequality holds

$$\nu_p(m_n(E)) \leq C_p(E,G) \tag{3}$$

for every condenser (E, G) with $m_n(G) < m_n(\Omega)/2$.

For example, if $\Omega = \mathbb{R}^n$, then

$$\nu_p(s) = \omega_n^{p/n} n^{1-p/n} \left| \frac{p-n}{p-1} \right|^{p-1} \left| m_n(G)^{\frac{p-n}{n(p-1)}} - s^{\frac{p-n}{n(p-1)}} \right|^{1-p}$$

for $p \neq n$ and

$$\nu_n(s) = n^{n-1} \omega_n \left(\log \frac{m_n(G)}{s} \right)^{1-n}.$$

The second isocapacitary function

$$\pi_p : [0, m_n(\Omega)/2] \to [0, \infty], \quad p > n,$$

is given by

$$\pi_p(s) = \inf\{C_p(E, G) : E \text{ is a point in } G,\ G \subset \Omega,\ \text{and } m_n(G) \leq s\}$$

The function π_p is clearly non-increasing and the corresponding isocapacitary inequality is

$$\pi_p(m_n(G)) \leq C_p(E, G). \tag{4}$$

For instance, in the case $\Omega = \mathbb{R}^n$

$$\pi_p(s) = \omega_n^{p/n}\left(\frac{p-n}{p-1}\right)^{p-1}(n\, s)^{1-p/n}.$$

The isoperimetric function of Ω, denoted by $\lambda : [0, m_n(\Omega)/2] \to [0, \infty]$ is given by

$$\lambda(s) = \inf\big\{\mathcal{H}^{n-1}(\Omega \cap \partial E) : E \subset \Omega,\ s \leq m_n(E) \leq m_n(\Omega)/2\big\}. \quad (5)$$

Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

If $\Omega = \mathbb{R}^n$, then

$$\lambda(s) = \omega_n^{1/n}(n\,s)^{(n-1)/n}.$$

The definition of λ leads to the relative isoperimetric inequality on Ω which reads

$$\lambda(m_n(E)) \leq \mathcal{H}^{n-1}(\Omega \cap \partial E) \tag{6}$$

for every $E \subset \Omega$ with $m_n(E) \leq m_n(\Omega)/2$.

The isoperimetric function of an open subset of $\mathbb{R}^n$ was introduced by Vladimir Maz'ya in 1960s. Nowadays it plays an important role in analysis on manifolds.

The functions ν_p and π_p on one hand, and λ on the other hand, are related by the inequalities

$$\nu_p(s) \geq \Big(\int_s^{m_n(\Omega)/2} \frac{dr}{\lambda(r)^{p'}}\Big)^{1-p} \tag{7}$$

and

$$\pi_p(s) \geq \Big(\int_0^{s} \frac{dr}{\lambda(r)^{p'}}\Big)^{1-p} \tag{8}$$

for $s \in (0, m_n(\Omega)/2)$ with $p' = p/(p-1)$, which follows along the same lines as similar assertions in Vladimir Maz'ya's book on Sobolev spaces.

Clearly,

$$\lambda(s) \le c_n\, s^{(n-1)/n}. \tag{9}$$

Moreover, if Ω is bounded and Lipschitz, then

$$\lambda(s) \approx s^{(n-1)/n} \quad \text{near } s = 0. \tag{10}$$

Furthermore, for any Ω,

$$\nu_p(s) \le c_{n,p}\, s^{(n-p)/n} \quad \text{for } n > p, \tag{11}$$

$$\nu_p(s) \le c_{n,p}(\Omega) \qquad \text{for } n < p, \tag{12}$$

and

$$\nu_p(s) \le c_n \left(\log \frac{1}{s}\right)^{1-n} \quad \text{for } n = p. \tag{13}$$

These inequalities can be verified by setting appropriate test functions in the definition of the p-capacity.

If Ω is bounded and Lipschitz, then, by (10) and (7),

$$\nu_p(s) \approx \begin{cases} s^{(n-p)/n} & \text{if } n > p, \\ \left(\log \frac{1}{s}\right)^{1-n} & \text{if } n = p \end{cases} \tag{14}$$

and

$$\pi_p(s) \approx s^{(n-p)/n} \qquad \text{for } p > n. \tag{15}$$

near $s = 0$.

Main results for arbitrary Ω

Theorem

For every $\varepsilon \in (0, m_n(\Omega)/2)$ *and for all* $u \in L^{1,p}(\Omega) \cap L^{1,r}(\Omega)$, $p > n$, $r \geq 1$,

$$\operatorname{osc}_{\Omega} u \leq \pi_p(\varepsilon)^{-1/p} \|\nabla u\|_{L^p(\Omega)} + \nu_r(\varepsilon)^{-1/r} \|\nabla u\|_{L^r(\Omega)}, \qquad (16)$$

where

$$\operatorname{osc}_{\Omega} u = \operatorname{ess}\sup_{\Omega} u - \operatorname{ess}\inf_{\Omega} u.$$

Corollary

Let $p \geq 1$, $r \geq 1$. Then, for every $\varepsilon \in (0, m_n(\Omega)/2)$ and for all $u \in L^{1,p}(\Omega) \cap L^{1,r}(\Omega)$

$$\operatorname{osc}_{\Omega} u \leq \Big(\int_0^{\varepsilon} \frac{d\mu}{\lambda(\mu)^{p'}}\Big)^{1/p'} \|\nabla u\|_{L^p(\Omega)} + \Big(\int_{\varepsilon}^{m_n(\Omega)/2} \frac{d\mu}{\lambda(\mu)^{r'}}\Big)^{1/r'} \|\nabla u\|_{L^r(\Omega)}. \tag{17}$$

Remark. One can add the inequality $p > n$ in Theorem since (16) has no sense for $n \geq p$.

In the case $r = 1$, $p > n$, the estimate (17) is simplified

$$\operatorname{osc}_{\Omega} u \leq \Big(\int_0^{\varepsilon} \frac{d\mu}{\lambda(\mu)^{p'}}\Big)^{1/p'} \|\nabla u\|_{L^p(\Omega)} + \lambda(\varepsilon)^{-1} \|\nabla u\|_{L^1(\Omega)}. \tag{18}$$

Domains of the class $\mathcal{J}_\alpha$

A domain belongs to the class $\mathcal{J}_\alpha$, $\alpha > 0$, if there is a constant $\mathcal{K}_\alpha$ such that

$$\lambda(\mu) \geq \mathcal{K}_\alpha \, \mu^\alpha$$

for $\mu \in (0, m_n(\Omega)/2)$. This class was introduced and studied in detail by Vladimir Maz'ya (1960). In particular, any Lipschitz domain is in $\mathcal{J}_{(n-1)/n}$.

Corollary

Let $\Omega \in \mathcal{J}_\alpha$, let $p > n$, $r \geq 1$ and let

$$\frac{1}{p'} > \alpha > \frac{1}{r'}.$$

Then, for every $\varepsilon \in (0, m_n(\Omega)/2)$ and for all $u \in L^{1,p}(\Omega)$

$$\begin{aligned} \operatorname{osc}_\Omega u \; &\leq \mathcal{K}_\alpha^{-1} \Big((1-\alpha p')^{-1/p'} \varepsilon^{-\alpha+1/p'} \|\nabla u\|_{L^p(\Omega)} \\ &+ (\alpha r' - 1)^{-1/r'} \varepsilon^{-\alpha+1/r'} \|\nabla u\|_{L^r(\Omega)} \Big). \end{aligned} \tag{19}$$

Taking the minimum value of the right-hand side in ε, we arrive at the multiplicative inequality

$$\mathrm{osc}_{\Omega} u \leq c_{\alpha,p,r}\, \mathcal{K}_{\alpha}^{-1} \|\nabla u\|^{\gamma}_{L^p(\Omega)} \|\nabla u\|^{1-\gamma}_{L^r(\Omega)} \qquad (20)$$

where Ω is a domain of the class $\mathcal{J}_{\alpha}$ with $1/p' > \alpha > 1/r'$, $p > n$, $r \geq 1$, and

$$\gamma = \frac{\alpha - 1/r'}{1/r - 1/p}.$$

The case $\Omega \in \mathcal{J}_{1/r'}$

We turn to the critical case when Ω belongs to the class $\mathcal{J}_{1/r'}$.

Corollary

Let $p > n$, $p > r \geq 1$ and let $\Omega \in \mathcal{J}_{1/r'}$. Then, for every $\varepsilon \in (0, m_n(\Omega)/2)$ and for all $u \in L^{1,p}(\Omega)$,

$$\operatorname{osc}_\Omega u \leq \mathcal{K}_{1/r'}^{-1}\Big(\Big(\frac{(p-1)r}{p-r}\Big)^{1/p'} \varepsilon^{(p-r)/pr} \|\nabla u\|_{L^p(\Omega)} + \Big(\log \frac{m_n(\Omega)}{2\varepsilon}\Big)^{1/r'} \|\nabla u\|_{L^r(\Omega)}\Big). \tag{21}$$

Without taking into account the values of the constants, we can write

$$\operatorname{osc}_\Omega u \le c\Big(\varepsilon^{(p-r)/pr}\|\nabla u\|_{L^p(\Omega)} + \Big(\log\frac{m_n(\Omega)}{2\varepsilon}\Big)^{1/r'}\|\nabla u\|_{L^r(\Omega)}\Big),$$

where $\varepsilon \in (0, m_n(\Omega)/2)$ and hence

$$\operatorname{osc}_\Omega u \le c_1\big(1 + \big|\log(c_2\|\nabla u\|_{L^p(\Omega)})\big|\big)^{1/r'}, \tag{22}$$

provided $\|\nabla u\|_{L^r(\Omega)} = 1$. Recall that $p > n$, $p > r \ge 1$, and $\Omega \in \mathcal{J}_{1/r'}$.

In particular, if Ω is a bounded Lipschitz domain, then $r = n$ and (22) becomes

$$\operatorname{osc}_\Omega u \le c_1\big(1 + \big|\log(c_2\|\nabla u\|_{L^p(\Omega)})\big|\big)^{(n-1)/n}, \tag{23}$$

with $\|\nabla u\|_{L^n(\Omega)} = 1$ (Brezis-Gallouet inequality).

Higher order Sobolev spaces

Theorem

Let $\Omega \in \mathcal{J}_\alpha$, $\alpha < 1$, *and let* u *denote an arbitrary function in* $W^{l,q}(\Omega)$ *with integer* l *and* $q \geq 1$. *Further let* $r = 1/(1-\alpha)$ *and*

$$l(1-\alpha) < 1/q. \tag{24}$$

If

$$\|u\|_{W^{1,r}(\Omega)} = 1,$$

then

$$\|u\|_{L^\infty(\Omega)} \leq C_{\alpha,q,l}\Big(1 + \big(\log(1 + \|u\|_{W^{l,q}(\Omega)})\big)^\alpha\Big). \tag{25}$$

Whirlpool domain

Example 1. Let Ω be the domain

$$\{x = (x', x_n) : |x'| < \varphi(x_n),\ 0 < x_n < 1\}, \tag{26}$$

where φ is a continuously differentiable convex function on $[0, 1]$, $\varphi(0) = 0$. The area minimizing function satisfies

$$c\,[\varphi(t)]^{n-1} \le \lambda\Big(v_{n-1}\int_0^t [\varphi(\tau)]^{n-1} d\tau\Big) \le [\varphi(t)]^{n-1} \tag{27}$$

for sufficiently small t. (V. Maz'ya, Sobolev spaces). Here v_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Now the inequality (17) implies

$$\operatorname{osc}_{\Omega} u \le c_1 \Big(\Big(\int_0^{\delta} \varphi(t)^{\frac{n-1}{p-1}} \, dt \Big)^{1/p'} \|\nabla u\|_{L^p(\Omega)} + c_2 \Big(\int_{\delta}^{1} \varphi(t)^{\frac{n-1}{r-1}} \, dt \Big)^{1/r'} \|\nabla u\|_{L^r(\Omega)} \Big) \tag{28}$$

for sufficiently small $\delta > 0$.

For the power β-cusp

$$\Omega = \left\{ x : \sum_{i=1}^{n-1} x_i^2 < x_n^{2\beta},\ 0 < x_n < 1 \right\}, \quad \beta > 1, \tag{29}$$

one has by (27)

$$c_1\, s^{\alpha} \le \lambda(s) \le c_2\, s^{\alpha}, \qquad \alpha = \frac{\beta(n-1)}{\beta(n-1)+1}.$$

For this particular case (28) takes the form

$$\operatorname{osc}_{\Omega} u \le c\left(\delta^{\left(1+\frac{\beta(n-1)}{p-1}\right)\frac{p-1}{p}} \|\nabla u\|_{L^p(\Omega)} + \delta^{\left(1+\frac{\beta(n-1)}{r-1}\right)\frac{r-1}{r}} \|\nabla u\|_{L^r(\Omega)} \right),$$

where $p - 1 + \beta(n-1) > 0$ and $r - 1 + \beta(n-1) < 0$.

In the critical case $r-1+\beta(n-1)=0$ one has for small $\delta>0$

$$\operatorname{osc}_{\Omega} u \leq c\left(\delta^{\left(1+\frac{\beta(n-1)}{p-1}\right)\frac{p-1}{p}}\|\nabla u\|_{L^p(\Omega)}+\left(\log\delta^{-1}\right)^{1/r'}\|\nabla u\|_{L^r(\Omega)}\right).$$

Minimizing the right-hand side in the preceding inequality, we arrive at inequality of Brezis-Wainger type for the β-cusp.

Theorem

Let Ω be the β-cusp (29) and let u denote an arbitrary function in $W^{l,q}(\Omega)$, where l is integer, $q \geq 1$, and

$$ql > \beta(n-1).$$

If

$$\|u\|_{W^{1,1+\beta(n-1)}(\Omega)} = 1,$$

then

$$\|u\|_{L^\infty(\Omega)} \leq C_{\beta,q,l}\Big(1 + \big(\log(1 + \|u\|_{W^{l,q}(\Omega)})\big)^{\alpha}\Big) \qquad (30)$$

with $\alpha = \beta(n-1)/[1 + \beta(n-1)]$.

Remark. We can show that the power α of the logarithm in (30) is the best possible by choosing

$$u(x) = \frac{\log \frac{1}{x_n+\delta}}{\left(\log \frac{1}{\delta}\right)^{1/[1+\beta(n-1)]}}$$

with a small $\delta > 0$.

λ-John domains

We recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is λ-John, $\lambda \geq 1$, if there is a constant $C > 0$ and a distinguished point $x_0 \in \Omega$ such that every $x \in \Omega$ can be joined to x_0 by a rectifiable arc $\gamma \subset \Omega$ along which

$$\mathrm{dist}(y, \partial\Omega) \geq C\,|\gamma(x,y)|^\lambda, \quad y \in \gamma,$$

where $|\gamma(x,y)|$ is the length of the portion of γ joining x to y.

Clearly, the class of λ-John domains increases with λ.

Kilpeläinen and Malý showed that every λ-John domain belongs to the class $\mathcal{J}_{\lambda(n-1)/n}$.

Our result implies that inequality

$$\|u\|_{L^\infty(\Omega)} \leq C_{\alpha,q,l}\Big(1 + \big(\log(1 + \|u\|_{W^{l,q}(\Omega)})\big)^{\alpha}\Big)$$

holds with $\alpha = \lambda(n-1)/n$ for every λ-John domain.