SPECTRAL ANALYSIS AND SPECTRAL SYNTHESIS IN THE SPACE OF TEMPERED FUNCTIONS ON DISCRETE ABELIAN GROUPS

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Let G be a locally compact abelian group (LCA–group), $\mathcal{F}$ be a locally covex topological vector space that cosists of complex-valued functions on G. The space $\mathcal{F}$ is called a translation invariant space if $\mathcal{F}$ is invariant under translations (shifts)

$$\tau_y : f(x) \mapsto f(x-y), \quad f(x) \in \mathcal{F}, y \in G, \tag{1}$$

and all operators τ_y are continuous operators on the space $\mathcal{F}$.
A closed linear subspace $\mathcal{H} \subseteq \mathcal{F}$ is called an invariant subspace if $\tau_y(\mathcal{H}) \subseteq \mathcal{H}$ for any $y \in G$.

A continuous homomorphism of G into the multiplicative group $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ of nonzero complex numbers is called an *exponential functions* or *generalized character* on G. A continuous homomorphism of G into the multiplicative group $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is called a character of G. Continuous homomorphisms of G into the additive group of complex numbers are called *additive* functions. A function $x \mapsto P(a_1(x), \ldots, a_m(x))$ on G is called a *polynomial* if $P(z_1, \ldots, z_m)$ is a complex polynomial in m variables and $a_1, \ldots, a_m$ are additive functions. A product of a polynomial and an exponential function is called an *exponential monomial*, and linear combinations of of exponential monomials are called *exponential polynomials*.

Let $\mathcal{F}$ be a translation invariant space on G, $\mathcal{H}$ be an invariant subspace in $\mathcal{F}$.

Definition 1

An invariant subspace $\mathcal{H}$ admits spectral synthesis if $\mathcal{H}$ coicides with the closed linear span in $\mathcal{F}$ of all exponential monomials that belong to $\mathcal{H}$. We say that a translation invariant space $\mathcal{F}$ has the spectral synthesis property if any invariant subspace $\mathcal{H} \subseteq \mathcal{F}$ admits spectral synthesis.

Definition 2

We say that spectral analysis holds in $\mathcal{F}$ if every nonzero invariant subspace $\mathcal{H} \subseteq \mathcal{F}$ contains an exponential function.

It can be proved that if a translation invariant space $\mathcal{F}$ has the spectral synthesis property than spectral analysis holds in $\mathcal{F}$.

§2. EXAMPLES OF SPECTRAL SYNTHESIS

1. $G = (\mathbb{R}, +)$

Any exponential monomial on $\mathbb{R}$ has the form $f(x) = P(x)\, e^{\lambda x}$, where $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $P(x)$ is a polynomial. The function spaces $C(\mathbb{R})$ of all continuous functions and $\mathcal{E}(\mathbb{R}) = C^{\infty}(\mathbb{R})$ of all infinitely differentiable functions have the spectral synthesis property. This is result of L. Schwartz:
[Sch1] L. Schwartz, *Théorie générale des fonctions moynne-périodiques,* Ann. of Math. (2) **48** (1947), 875–929.

Some other examples of functions spaces on $\mathbb{R}$ with spectral synthesis property was studied in the papers:

[Gil] Gilbert J. E., On the ideal structure of some algebras of analytic functions, Pacif. J. of Math. **35** (1978), no. 3, 625–639.
[Pl1] Platonov S. S., Spectral synthesis in some topological vector spaces of functions, St. Petersburg Math. J. **22** (2011), no. 5, 813–833.

2. $G = (\mathbb{R}^n, +)$, $n \geq 2$

Any exponential monomial on $\mathbb{R}^n$ has the form $f(x) = P(x)\, e^{\lambda x}$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, $\lambda x = \lambda_1 x_1 + \cdots + \lambda_n x_n$, $P(x)$ is a polynomial in x. In [Sch1] L. Schwartz conjectured that the spaces $C(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ have the spectral synthesis property. This conjecture was disproved by D. I. Gurevich who constructed in [Gur] an example of a nonzero invariant subspace $\mathcal{H} \subset \mathcal{E}(\mathbb{R}^2)$ containing no exponential monomials.

[Gur] D. I. Gurevich., Counterexamples to a problem of L. Schvartz, Funct. Anal. Appl. **9:2** (1975), 116–120.

Nevertheless, L. Schwartz [Sch2] proved that the space $\mathcal{S}'(\mathbb{R}^n)$ of all tempered ditributions on $\mathbb{R}^n$ has the spectral synthesis property.

[Sch2] Schwartz L., *Analyse et synthése harmonique dans les espaces de distributions*, Can. J. Math., **3** (1951), 503–512.

3. G is a discrete group

For the case when G is a discrete group, the most natural function space is the space $C(G)$ consisting of all complex-valued functions on G with the topology of pointwise convergence.
The case $G = \mathbb{Z}^n$ was studied by M. Lefranc [Lef]. He proved that the space $C(\mathbb{Z}^n)$ has the spectral synthesis property.
[Lef] Lefranc M., *Analyse spectrale sur Z_n.* C. R. Acad. Sci. Paris, **246** (1958), 1951–1953.

Some results about the spectral synthesis on the discrete groups was cosidered in [Sz1].

[Sz1] Székelyhidi L., *Discrete spectral synthesis and its applications*, Berlin: Springer, 2006.

In particular, the space $C(G)$ has the spectral synthesis property if G is a finitely generated Abelian group [Sz2] or a torsion Abelian group [Ber-Sz].

[Sz2] Székelyhidi L., *On discrete spectral synthesis*, Functional Equations — Results and Advances (Z. Daroćzy and Zs. Páles), Dordrecht: Kluwer Academic Publishers, 2002, p. 263–274.

[Ber-Sz] Bereczky A., Székelyhidi L., *Spectral synthesis on torsion groups*, J. Math. Anal. Appl., **304** (2005), 607–613.

Let $r_0(G)$ denote the torsion free rank of G; that is, the cardinality of a maximal independent system of elements of infinite order.
In [Lacz-Sz 1] M. Laczkovich and L. Székelyhidi proved that the spectral synthesis in the space $C(G)$ holds on a discrete Abelian group G if and only if the torsion free rank of G is finite.
[Lacz-Sz] Laczkovich M., Székelyhidi L., *Spectral synthesis on discrete Abelian groups*, Math. Proc. Cambr. Phil. Soc., **143** (2007), 103–120.
In [Lacz-Sz 2] M. Laczkovich and L. Székelyhidi proved that the spectral analysis holds in the space $C(G)$ on a discrete Abelian group G if and only if the torsion free rank of G is less than continuum.
[Lacz-Sz 2] Laczkovich M., Székelyhidi L. *Harmonic analysis on discrete Abelian groups*, Proc. of the Amer. Math. Soc. 2004. V. 133. P. 1581–1586.

The problem of description of all LCA-groups G that the spectral synthesis holds in the space $C(G)$ is open. In [Pl2] and [Pl3] it was proved that the spectral synthesis in the space $C(G)$ holds for any element-wise compact Abelian groups.

[Pl2] Platonov S. S. *On spectral synthesis on zero-dimensional Abelian groups*, Sbornik: Mathematics. 2013. V. 204:9. P. 1332–1346.
[Pl3] Platonov S. S. *On spectral synthesis on element-wise compact Abelian groups* , Sbornik: Mathematics. 2015. V. 206:8. P. 1150–1172.

§3. THE SPACE OF TEMPERED FUNCTIONS ON DISCRETE ABELIAN GROUPS

Other natural functional space is the space $\mathcal{S}'(G)$ of all tempered disributions on a LCA-group G. The definition of the space $\mathcal{S}'(G)$ was introduced by F. Bruhat in [Br] for every LCA-group G. As in classical case $G = \mathbb{R}$, the space $\mathcal{S}'(G)$ is a dual space to the space $\mathcal{S}(G)$ of rapidly decreasing Bruhat–Schwartz functions. The definition and main properties of the Bruhat–Schwartz functions see in [Br] and [Osb].

[Br] Bruhat F. *Distributions sur un groupe localement compact et applications à l'étudedes représentations des groupes p-adiques*, Bull. Soc. math. France, **89**, 1961, 43–75.
[Osb]Osborne M. S., *On the Schwartz – Bruhat space and Paley – Wiener theorem for locally compact Abelian groups*, J. of Funct. Anal., **19**, 1975, 40–49.

We study some problems of spectral analysis and synthesis in the space $\mathcal{S}'(G)$ for the case when G is a discrete Abelian group. In this case tempered distributions from $\mathcal{S}'(G)$ are usual functins, then we call the space $\mathcal{S}'(G)$ by space of tempered functions on G.

R. J. Elliot claimed in [Ell] that spectral synthesis holds for every discrete Abelian group G in the spaces $C(G)$ and $\mathcal{S}'(G)$. As it turned out, the Elliot's proof was defective. In the paper [Sz3] was costructed an exampe of discrete Abelian group G such that spectral synthesis fails in the space $C(G)$.

[Ell] Elliot R. J., *Two notes on spectral synthesis for discrete Abelian groups*, Proc. Camb. Phil. Soc., **61**, 1965, 617–620.
[Sz3] Székelyhidi L., *The failure of spectral synthesis on some types of discrete Abelian groups*, J. Math. Anal. Appl., **291**, 2004, 757–763.

Let G be a discrete Abelian group. In order to define the space of of tempered functions on G we will consider the special case when G is a finitely generated Abelian group in the first, and then we will pass to the general case when G is an arbitrary Abelian group.
Let G be a finitely generated Abelian group and $v_1, \dots, v_m$ be a system of generators of G. Any element $x \in G$ can be represented in the form $x = t_1 v_1 + \dots + t_n v_n$, $t_j \in \mathbb{Z}$, $j = 1. \dots, n$. We define the quasinorm $|x|$ on G by

$$|x| := \min\{|t_1| + \dots + |t_n| : x = t_1 v_1 + \dots + t_n v_n,\ t_j \in \mathbb{Z}\}.$$

For any $k > 0$ we denote by $\mathcal{S}'_k(G)$ the set of all complex-valued functions $f(x)$ on G that satisfy

$$|f(x)|\,(1+|x|)^{-k} \to 0 \quad \text{as} \quad |x| \to \infty.$$

The norm

$$\|f\|_k := \sup_{x\in G} |f(x)|\,(1+|x|)^{-k}$$

makes $\mathcal{S}'_k(G)$ a Banach space. We put

$$\mathcal{S}'(G) := \bigcup_{k>0} \mathcal{S}'_k(G)$$

and endow $\mathcal{S}'(G)$ with the topology of the inductive limit of the Banach spaces $\mathcal{S}'_k(G)$. We call $\mathcal{S}'(G)$ the space of tempered functions on G.

Let us consider the general case, when G is an arbitrary Abelian group. For any finite subset $P \subset G$ we denote by G_P the subgroup of G generated by P. As the group G_P is finitely generated than the space $\mathcal{S}'(G_P)$ is already defined. For any function f on G let $\sigma_P(f) = f|_{G_P}$ be the restriction of f on G_P. By definition, the space $\mathcal{S}'(G)$ consists from all functions f on G such that $\sigma_P(f) \in \mathcal{S}'(G_P)$ for any finite subset $P \subset G$. The space $\mathcal{S}'(G)$ is a locally convex topological vector space with the topology of projective limit of the locally convex spaces $\mathcal{S}'(G_P)$.

§4. THE MAIN RESULTS

Theorem 1

Spectral analysis holds in the space $\mathcal{S}'(G)$ for any discrete Abelian group G.

We note that in [Lacz-Sz 2] M. Laczkovich and L. Székelyhidi proved that the spectral analysis in the space $C(G)$ holds on a discrete Abelian group G if and only if the torsion free rank of G is less than the continuum.

[Lacz-Sz 2] Laczkovich M., Székelyhidi L. *Harmonic analysis on discrete Abelian groups*, Proc. of the Amer. Math. Soc. 2004. V. 133. P. 1581–1586.

Theorem 2

Spectral synthesis fails in the space $\mathcal{S}'(G)$ if the torsion free rank $r_0(G)$ is infinite.

Therefore, the spectral synthesis does not hold in the space $\mathcal{S}'(G)$ for any discrete Abelian group G, and Elliot's statement is falls.

The natural problem: for which Abelian groups G the spectral synthesis hold in the space $\mathcal{S}'(G)$? The next theorem gives some sufficient condition.

Theorem 3

Spectral synthesis holds in the space $\mathcal{S}'(G)$ if G is a finitely generated Abelian group.

We note that the condition $r_0(G) < \infty$ is a necessary condition for the spectral synthesis property in the space $\mathcal{S}'(G)$. The open problem: is the condition $r_0(G) < \infty$ is a sufficient condition for the spectral synthesis property in the space $\mathcal{S}'(G)$?