Classical singular operators on integers and their L^p norms

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Basic definitions, classical case

The Hilbert transform on the real line is defined by

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} f(y)\frac{1}{x-y}dy$$

which is

$$\widehat{Hf}(\xi) = -i\frac{\xi}{|\xi|}\hat{f}(\xi)$$

and

$$H \circ \sqrt{-\Delta} = \partial.$$

In $\mathbb{R}^N$ the Riesz transforms are

$$R^i \circ \sqrt{-\Delta} = \partial_i$$

Basic definitions, discrete case

In this lecture, we are interested in Riesz transforms on products of discrete abelian groups of a single generator 1, for example $\mathbb{Z}^N$.

Discrete first derivatives:

$$\partial^i_+ f(n) = f(n+1_i) - f(n)$$

and

$$\partial^i_- f(n) = f(n) - f(n-1_i)$$

Discrete Laplace:

$$\Delta f(n) = \sum_{i=1}^{N} \partial^i_+ \partial^i_- f(n) = \sum_{i=1}^{N} [f(n+1_i) - 2f(n) + f(n-1_i)]$$

There are two choices of Riesz transforms for each direction

$$R^i_\pm \circ \sqrt{-\Delta} = \partial^i_\pm$$

Second order Riesz transforms

The N second order discrete Riesz transforms are

$$R_i^2 = R_+^i R_-^i$$

We are concerned with operators of the form

$$R_\alpha^2 = \sum_{i=1}^{N} \alpha_i R_i^2$$

where $|\alpha_i| \leqslant 1$.

Classical Case

In the classical situation, $\mathbb{R}^2$, this includes

$$R_1^2 - R_2^2 = \mathrm{Re}B$$

where B is the Beurling Ahlfors operator. The following estimate is due to Nazarov and Volberg:

$$\|\mathrm{Re}B\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \leqslant p^* - 1$$

Here $p^* - 1 = \max\left\{p-1, \frac{1}{p-1}\right\}$
Sharpness is due to Geiss, Montgomery-Smith, Saksman.

On discrete Abelian groups

Theorem (Domelevo, P. (2014))

If $\|\alpha\|_\infty \leqslant 1$ *and* $G = \mathbb{Z}$ *or* $G = \mathbb{Z}/m\mathbb{Z}$ *then*

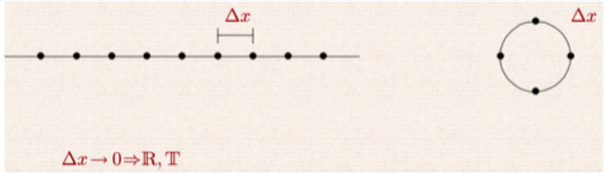
$$\|R^2_\alpha\|_{L^p(G^N)\to L^p(G^N)} \leqslant p^* - 1 = \max\left\{p-1, \frac{1}{p-1}\right\}$$

The estimate is sharp for products of infinite groups and sharp for products of finite groups if one requires a uniform estimate that holds for all orders m.

In the real valued case and when $0 \leqslant a \leqslant \alpha_i \leqslant b \leqslant 1 \forall i$ the estimate can be improved to the so called Choi constants. They are better than $p^* - 1$ but less explicit. This case includes all single second order Riesz transforms R^2_i. Here $a = 0$ and $b = 1$.

Discretizations and Finite Difference Schemes

$\mathbb{Z}$ is a special and very regular discretization of $\mathbb{R}$ while $\mathbb{Z}/m\mathbb{Z}$ discretizes $\mathbb{T}$.



In the finite difference scheme with the mesh and derivatives defined as above, the discrete Riesz transforms can be regarded as a finite difference approximation of classical Riesz transforms.

By considering finer meshes, we see that our estimates recover the Nazarov-Volberg estimate (but not vice versa) and that we inherit sharpness from that of the respective continuous settings.

What we know: Sharp L^pestimates.

Two prototypes, two functions:

- periodic Hilbert transform

- differentially subordinate martingales

In both cases, the estimates are obtained by the discovery of a special function of several variables that is characteristic in some sense for the problem.

Hilbert transform (Pichorides)

Theorem (Pichorides)

Let f be 2π periodic and $\tilde{f}$ its conjugate function. Then the best constants in Riesz's theorem are

$$\|\tilde{f}\|_p \leqslant A_p \|f\|_p$$

where $A_p = \tan\left(\frac{\pi}{2p}\right)$ *when* $1 < p \leqslant 2$ *and* $\cot\left(\frac{\pi}{2p}\right)$ *when* $p > 2$.

Essén's proof.

Let f be harmonic on the disk and $\tilde{f}$ its conjugate function with $\tilde{f}(0) = 0$. So that $F = f + i\tilde{f}$ analytic.

Suppose $1 < p < 2$ and we have a function $G : \mathbb{C} \to \mathbb{R}$ with the following properties:

- $G(x) \leqslant 0$ for $x \in \mathbb{R}$
- $G(z)$ superharmonic in $\mathbb{C}$
- $G(z) \geqslant |z|^p - \cos^{-p}\left(\frac{\pi}{2p}\right)|x|^p$ for $z \in \mathbb{C}$ where $x = \mathrm{Re} z$

then plug $F(re^{i\varphi})$ into G and integrate over φ:

$$|F(re^{i\varphi})|^p \leqslant \cos^{-p}\left(\frac{\pi}{2p}\right)|f(re^{i\varphi})|^p + G(F(re^{i\varphi}))$$

It is then easy to pass to Pichorides estimate.

Essén's function

$$G(z) =$$

$$\begin{array}{lll} |z|^p - \cos^{-p}\left(\frac{\pi}{2p}\right)|x|^p & \text{when} & \frac{\pi}{2p} < |\arg(z)| < \pi - \frac{\pi}{2p} \\ -\tan\left(\frac{\pi}{2p}\right)|z|^p \cos(p\arg(z)) & \text{when} & |\arg(z)| < \frac{\pi}{2p} \\ -\tan\left(\frac{\pi}{2p}\right)|z|^p \cos(p(\pi - |\arg(z)|)) & \text{when} & 0 \leqslant \pi - |\arg(z)| < \frac{\pi}{2p} \end{array}$$

Burkholder's Estimate

Theorem (Burkholder)

$(\Omega, \mathfrak{F}, \mathbb{P})$ *probability space with filtration* $\mathfrak{F} = (\mathfrak{F}_n)_{n\in\mathbb{N}}$. *X and Y complex valued martingales with differential subordination:*

$$|Y_0(\omega)| \leqslant |X_0(\omega)|$$

$$|Y_n(\omega) - Y_{n-1}(\omega)| \leqslant |X_n(\omega) - X_{n-1}(\omega)|$$

a.s. in Ω. *Then*

$$\|Y\|_p \leqslant (p^* - 1)\|X\|_p, 1 < p < \infty$$

Dyadic martingale in $[0,1]$

Dyadic system

$$\mathcal{D} = \{[l2^{-k}, (l+1)2^{-k}] : 0 \leqslant l < 2^k, k \geqslant 0\}$$

$\{h_I : I \in \mathcal{D}\}$ is an orthonormal basis in $L^2([0,1])$ and so

$$f(x) = \int_0^1 f(t)dt + \sum_{I \in \mathcal{D}} (f, h_I) h_I(x)$$

When taking only ‘large’ intervals, obtain approximations of f.

Burkholder's estimate: weak form

For a fixed $f : [0,1] \to \mathbb{C}$ and dyadic system $h_I : I \in \mathcal{D}$ with sequence $|\sigma_I| = 1$ build a pair of differentially subordinate martingales

$$X_0 = \int_0^1 f(t)dt \text{ and } X_n = \sum_{I \in \mathcal{D}, |I| \geqslant 2^{-n}} (f, h_I) h_I$$

$$Y_0 = \int_0^1 f(t)dt \text{ and } Y_n = \sum_{I \in \mathcal{D}, |I| \geqslant 2^{-n}} \sigma_I (f, h_I) h_I$$

Differential subordination: $Y_0 = X_0$ and $|Y_n - Y_{n-1}| = |X_n - X_{n-1}|$ pointwise.

Burkholder's estimate: weak form

With $T_\sigma f = \sum_{I \in \mathcal{D}} \sigma_I (f, h_I) h_I$, Burkholder's theorem asserts that

$$\sup_\sigma \| T_\sigma \|_{p \to p} \leqslant p^* - 1$$

In its weak form, this becomes

$$\sup_\sigma |(T_\sigma f, g)| \leqslant (p^* - 1) \|f\|_p \|g\|_q$$

or by choosing the worst σ:

$$\sum_{I \in \mathcal{D}} |(f, h_I)(g, h_I)| \leqslant (p^* - 1) \|f\|_p \|g\|_q$$

Bellman function

Burkholder's theorem in its weak form

$$\sum_{I\in\mathcal{D}} |(f,h_I)(g,h_I)| \leqslant (p^*-1)\|f\|_p\|g\|_q$$

takes the localized form

$$\frac{1}{4|J|}\sum_{I\in\mathcal{D},I\subseteq J} |I||\Delta_I f||\Delta_I g| \leqslant (p^*-1)\langle |f|^p\rangle_J^{1/p}\langle |g|^q\rangle_J^{1/q}$$

where $\langle h\rangle_I = \frac{1}{|I|}\int_I h(t)dt$ mean value of h over I and $\Delta_I h = \langle h\rangle_{I_+} - \langle h\rangle_{I_-}$ the dyadic derivative.

Bellman function

The estimate

$$\frac{1}{|J|}\sum_{I\in\mathcal{D},I\subseteq J}\frac{1}{4}|I||\Delta_I f||\Delta_I g| \leqslant (p^*-1)\langle|f|^p\rangle_J^{1/p}\langle|g|^q\rangle_J^{1/q}$$

is a statement about relations of

$$\mathbf{f}=\langle f\rangle_I, \mathbf{g}=\langle g\rangle_I, \mathbf{F}=\langle|f|^p\rangle_I, \mathbf{G}=\langle|g|^q\rangle_I$$

Clearly $|\mathbf{f}|^p \leqslant \mathbf{F}$ and $|\mathbf{g}|^q \leqslant \mathbf{G}$.

Bellman function

By setting up a natural extremal problem, Burkholder's estimate implies the existence of a function B defined on the domain

$$D_p = \{(\mathbf{F}, \mathbf{G}, \mathbf{f}, \mathbf{g}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C} : |\mathbf{f}|^p \leqslant \mathbf{F}, |\mathbf{g}|^q \leqslant \mathbf{G}\}$$

with range

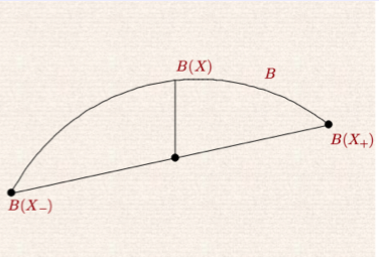
$$0 \leqslant B(\mathbf{F}, \mathbf{G}, \mathbf{f}, \mathbf{g}) \leqslant (p^* - 1)\mathbf{F}^{1/p}\mathbf{G}^{1/q}$$

and convexity

$$-d^2 B(\mathbf{F}, \mathbf{G}, \mathbf{f}, \mathbf{g}) \geqslant 2|d\mathbf{f}||d\mathbf{g}|$$

Bellman function

Midpoint concavity is equivalent to concavity.



$$-d^2 B(x,y) \geq 2|dx||dy|$$

iff

$$B(x,y) - \frac{1}{2}B(x_+,y_+) - \frac{1}{2}B(x_-,y_-) \geq \frac{1}{4}|\Delta x||\Delta y|$$

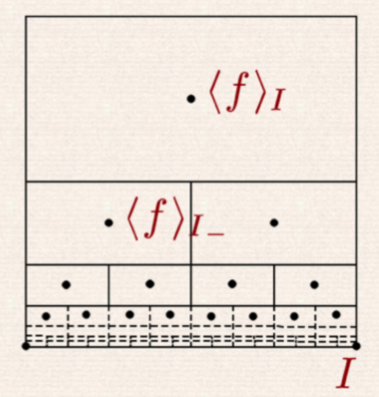
Bellman function

Infact, any such function proves the dyadic version of Burkholder's theorem.

$$\begin{aligned}
|J|(p^*-1)\langle|f|^p\rangle_J^{1/p}\langle|g|^q\rangle_J^{1/q} &\geqslant |J|B(v_J) \geqslant |J_+|B(v_{J_+}) + |J_-|B(v_{J_-}) \\
&\quad + \frac{1}{4}|J||\Delta_J f||\Delta_J g| \\
&\geqslant \ldots \\
&\geqslant \sum_{I\subset J, |I|=2^{-n}|J|} |I|B(v_I) \\
&\quad + \frac{1}{4}\sum_{I\subset J, |I|>2^{-n}|J|} |I||\Delta_I f||\Delta_I g|
\end{aligned}$$

Dyadic heat extension

$$2\langle f\rangle_I = \langle f\rangle_{I_+} + \langle f\rangle_{I_-}$$



Fourier Analysis

$$\widehat{\ }: (f : \mathbb{Z}^N \to \mathbb{C}) \to (\hat{f} : \mathbb{T}^N \to \mathbb{C})$$

$$f(\vec{n}) \mapsto \hat{f}(\vec{\xi}) = \sum_{\vec{n} \in \mathbb{Z}^N} f(\vec{n}) e^{-2\pi i \vec{n} \cdot \vec{\xi}}$$

$\partial^j_\pm, \Delta, R^i_\pm$ are multiplier operators.
For example

$$\widehat{R_j^2} = \widehat{R^j_+ R^j_-} = \frac{-4\sin^2(\pi\xi_j)}{4\sum_i \sin^2(\pi\xi_i)}$$

Heat Extension

$f : \mathbb{Z}^N \to \mathbb{C}$. Its heat extension is $\tilde{f} : \mathbb{Z}^N \times [0, \infty) \to \mathbb{C}$ with

$$\tilde{f}(t, \vec{n}) = e^{t\Delta} f(\vec{n})$$

Using basic Fourier analysis, semigroups or integration by parts, one derives the formula: if $\hat{g}(0) = 0$ then

$$(f, R_j^2 g) = -2 \int_0^\infty \sum_{\mathbb{Z}^N} \partial_+^j \tilde{f}(\vec{n}, t) \partial_+^j \tilde{g}(\vec{n}, t) dt$$

and observes its similarity to the model

$$(T_\sigma f, g) = \sum_{I \in \mathcal{D}} \sigma_I (f, h_I)(g, h_I) = \frac{1}{4} \sum_{I \in \mathcal{D}} \sigma_I (\langle f \rangle_{I_+} - \langle f \rangle_{I_-})(\langle g \rangle_{I_+} - \langle g \rangle_{I_-}).$$

$|I| \sim t$ dyadic heat extension vs heat extension, space derivatives

Bellman transference

Instead of

$$b(I) = B(\langle |f|^p \rangle_I, \langle f \rangle_I, \langle |g|^q \rangle_I, \langle g \rangle_I)$$

evaluate

$$b(\vec{n}, t) = B(\widetilde{|f|^p}, \tilde{f}, \widetilde{|g|^q}, \tilde{g})(\vec{n}, t)$$

$b(I) - \frac{1}{2}b(I_+) - \frac{1}{2}b(I_+)$ estimate becomes $(\partial_t - \Delta)b$.

For classical heat equations, say in (x, t) this transference works perfectly:

$$(\partial_t - \Delta)b = (-d^2B(\tilde{v})\tilde{v}'_x, \tilde{v}'_x)$$

because $\tilde{v}$ solves the heat equation.

Bellman transference

In the discrete case: lack of chainrule.

Very easy way out:

$$-d^2B(v) \geq 2|dx||dy|$$

where the right hand side is independent of the location v of evaluation of Hessian. The integral form of the remainder in Taylor theorem connecting two jumps uses this $2|dx||dy|$ estimate.

This simplicity is also seen in the probabilistic proof we present now and becomes 'strong subordination'.

Probability for Analysts 1

$[0, 1]$ with Lebesgue measure is a probability space.

For each discrete time $0 \leq n$ consider the classical dyadic covering of size 2^{-n}, together with its generated sigma algebra.
This becomes a filtered probability space $([0, 1], F, dx)$

$$X_n = E(f|F_n) \text{ and } Y_n = E(T_\sigma f|F_n)$$

are a pair of martingales that are differentially subordinate.

Indeed, $E(f|F_n) = \sum_{1\leq k\leq n} \Delta_k(f) + E(f)$. So

$$|\Delta_n(T_\sigma f)| \leq |\sigma_n||\Delta_n(f)|$$

Here, $dX_n = \Delta_n(f) = E(f|F_n) - E(f|F_{n-1})$
Note σ_n is measurable in F_n, such multipliers are called predictable.

Probability for Analysts 2

Square bracket, discrete filtration:

$$[X, X]_n = \sum_{k=1}^{n} (dX_k)^2$$

for example discrete random walk B: $[B, B]_n = \sum_{k=1}^{n} 1 = n$

Modern probability theory is concerned with filtered probability spaces with continuous time: (Ω, F, μ), for example the Brownian filtration.

Probability for Analysts 3

Square bracket and products (almost surely):

$$(XX)_n - (XX)_{n-1} = 2X_{n-1}(X_n - X_{n-1}) + (X_n - X_{n-1})^2 = 2X_{n-1}dX_n + (dX_n)^2$$

This can be generalised to continuous in time filtrations and defined the square bracket as the predictable compensator of the product of martingales. One obtains the bracket process:

$$[X, X] = X^2 - \int X_- dX$$

Indeed, $X_0^2 + \sum_i (X^{T_{i+1}^n} - X^{T_i^n})$ with T^n for all n sequence of increasing stopping times.

Differential subordination

Y differentially subordinate to X if $[X, X]_t - [Y, Y]_t$ is a non-negative and non-decreasing function of $t \geq 0$.

If the martingale has discontinuous paths (jumps) then this bracket differs from $\langle , \rangle$ in that subordination gives precise information at the instances of jumps.

Why is this related to CZO?

There is the famous formula of Gundy Varopoulos. There is also one for jumps (second order Riesz transforms) with strong subordination

Theorem (Arcozzi, Domelevo, P. (2015))

The second order Riesz transforms $R_i^2 f$, $1 \leq i \leq m$, and $R_{jk} f$, $1 \leq j, k \leq n$, of a function $f \in L^2(G)$ as defined in can be written as the conditional expectations

$$R_i^2 f(z) = \mathbb{E}(M^{i,f} | \mathcal{Z}_0 = z) \text{ and } R_{jk} f(z) = \mathbb{E}(M^{j,k,f} | \mathcal{Z}_0 = z).$$

Here $M_t^{i,f}$ and $M_t^{j,k,f}$ are suitable martingale transforms of the martingale M_t^f associated to f, and $\mathcal{Z}_t$ is a suitable random walk on G

$M_t^{\alpha,f,T,\mathcal{Z}_0} = f(T, \mathcal{Z}_0) + \int_0^t (A_\alpha \nabla_z f(s, \mathcal{Z}_{s_-}), d\mathcal{Z}_s)$ using the augmented gradient $\nabla f = (X_1^+, ..., X_n^+, X_1^-, ..., X_n^-, Y_1, ..., Y_m)$

The discrete Hilbert transform

We consider the averages version of the discrete Hilbert:

$$\frac{1}{2}(H_+ + H_-)$$

Its kernel can be calculated and its Fourier multiplier is a signum cosinus, square of operator is not quite $-I$.

The underlying weak formulation for the Hilbert transforms involves Poisson extensions instead of heat extensions, this means that time is also governed by a Brownian motion instead of deterministic.

Furthermore, the formula requires a 4 by 4 'rotation' matrix, treating continuous time t as if it were discrete.

Differential subordination and orthogonality only with respect to sharp bracket.

Riesz vector estimates

Some of these difficulties are also visible if one looks for dimensionless estimates of Riesz vector:

Mid to late 90s:

- Riesz transforms in $\mathbb{R}^N$: Iwaniec, Martin.
- Riesz transforms on compact Lie groups: Arcozzi.
- certain orthogonal, differentially subordinate martingales: Banuelos, Wang.

The L^p norms of the discrete Hilbert transform(s) is a famous open question.

What we know: dimensional behavior in L^p of Riesz vector

The square function of the Riesz vector or ℓ^2 of the Riesz vector $f \mapsto |\overrightarrow{R_i f}|_{\ell^2}$ has dimensionless L^p bounds in

- $\mathbb{R}^N$ (Stein/Pisier/Dragicevic, Volberg)
- Gaussian setting (Meyer/Pisier/Dragicevic, Volberg)
- Heisenberg group (Coulhon, Mueller, Zienkiewicz/Piquard)
- Riemannian manifolds (Carbonaro, Dragicevic)

What we know: dimensional behavior in L^p of Riesz vector

Francoise Piquard:

In $\mathbb{Z}^N$ this dimension-free behavior is only seen for $p \geqslant 2$ and there is a dimensional growth when $1 < p < 2$.

Positive result: non-commutative methods.

Negative result: uses the fact that functions can have non-zero derivatives outside of their support.

Lamberton's/Piquard's example

We try to arrive at a contradiction for

$$\|\overrightarrow{\partial_+^k F}\|_{L^p(\ell^2)} \leqslant C_p \|(-\Delta)^{1/2} F\|_{L^p}$$

with C_p independent of dimension.

To construct test functions F, choose tensor products made of $f : \mathbb{Z} \to \mathbb{R}$ supported in $A = \{-1, 1\}$:
$f = -\mathcal{X}_{\{-1\}} + \mathcal{X}_{\{1\}}$ (nearly any mean 0 function will do)

$$F : \mathbb{Z}^N \to \mathbb{R}, x \mapsto \prod_{j=1}^{N} f(x_j)$$

Then $\|F\|_{L^p} = \|f\|_{L^p}^N$ and $\partial_+^k F(x) = \partial_+^k f(x_k) \prod_{j \neq k} f(x_j)$.

Lamberton's/Piquard's example

Thus

$$\sqrt[p]{N}\|1_{A^c}\partial_+ f\|_{L^p} \leqslant C_p\sqrt{N}\|\partial_+\partial_- f\|_{L^p}^{1/2}\|f\|_{L^p}^{1/2}.$$

When $\|1_{A^c}\partial_+ f\|_{L^p} \neq 0$ this is impossible for $p < 2$. Recall $f = -\mathcal{X}_{\{-1\}} + \mathcal{X}_{\{1\}}$ and $A = \{-1, 1\}$ and thus $\partial_+ f$ has support outside of A.