Spectrally reasonable measures

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Definitions

$M(\mathbb{T})$ – the Banach algebra of complex-valued, Borel regular measures on the circle group with the usual convolution as multiplication.
For $\mu \in M(\mathbb{T})$ we define the spectrum $\sigma(\mu)$ of μ as the set

$$\sigma(\mu) = \{\lambda \in \mathbb{C} : \mu - \lambda\delta_0 \text{ is not invertible}\}.$$

It follows from the general theory of commutative Banach algebras that the spectrum of a measure is an image of its Gelfand transform and it is non-empty compact subset of the complex plane. The Gelfand space of the measure algebra on the circle group will be denoted by $\triangle(M(\mathbb{T}))$.
For $\mu \in M(\mathbb{T})$ we also define the n-th Fourier-Stieltjes coefficient:

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{-int} d\mu(t).$$

Formulation of the problem

Main problem

It is obvious that for every $\mu \in M(\mathbb{T})$ we have $\overline{\widehat{\mu}(\mathbb{Z})} \subset \sigma(\mu)$. But when do we have $\overline{\widehat{\mu}(\mathbb{Z})} = \sigma(\mu)$?

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Natural spectrum

Following M. Zafran we say that a measure $\mu \in M(\mathbb{T})$ has a natural spectrum iff $\overline{\widehat{\mu}(\mathbb{Z})} = \sigma(\mu)$. The set of all such measures will be denoted by $\mathcal{N}$.

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Examples

Absolutely continuous and purely discrete measures have a natural spectrum.

Bad behaviour of measures

The Wiener-Pitt phenomenon

There exists a measure $\mu \in M(\mathbb{T})$ for which $\overline{\widehat{\mu}(\mathbb{Z})} \neq \sigma(\mu)$.

There are few proofs of the existence of the Wiener-Pitt phenomenon (usually using using Riesz products or measures supported on special 'thin' sets) and it appears that the spectrum of a measure can be much bigger than the closure of the set of values of the Fourier-Stieltjes transform.

Example: Riesz product

For the classical Riesz product

$$\mu := \prod_{k=1}^{\infty} (1 + \cos(4^k t))$$

we have the formula for the Fourier-Stieltjes coefficients:

$$\widehat{\mu}\left(\sum_{k=1}^{n} \varepsilon_k 4^k\right) = \prod_{k=1}^{n} \left(\frac{1}{2}\right)^{|\varepsilon_k|} \text{ where } \varepsilon_k \in \{-1, 0, 1\} \text{ and}$$

$$\widehat{\mu}(m) = 0 \text{ if } m \text{ is not expressible in the above form}$$

and so

$$\overline{\widehat{\mu}(\mathbb{Z})} = \{0\} \cup \left\{\frac{1}{2^n}\right\}_{n\in\mathbb{N}} \cup \{1\}$$

but it can be proved that $\sigma(\mu) = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Properties of the set of all measures with a natural spectrum

The set $\mathcal{N}$ has the following properties:

1. It is closed in the norm topology and under involution.
2. It is closed under the action of functional calculus.

But, as it was observed by M. Zafran, the set $\mathcal{N}$ is not closed under addition. In fact, the following theorem holds true:

Theorem of Hatori and Sato

$$\mathcal{N} + \mathcal{N} + M_d(\mathbb{T}) = M(\mathbb{T}).$$

Here $M_d(\mathbb{T})$ is the subalgebra of discrete (purely discrete measures).

Spectrally reasonable measures

Since the set $\mathcal{N}$ is not closed under addition we will investigate the set of all suitable perturbations.

Reasonability

We say that a measure $\mu \in M(\mathbb{T})$ is spectrally reasonable if for all $\nu \in \mathcal{N}$ we have $\mu + \nu \in \mathcal{N}$. The set of all spectrally reasonable measures will be denoted by $\mathcal{S}$.

It is obvious that $\mathcal{S} \subset \mathcal{N}$ and $\mathcal{S}$ is a closed linear space. However, a much stronger fact, requiring a lot of additional algebraic manipulations, is true:

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Theorem

The set $\mathcal{S}$ is a symmetric Banach *-subalgebra of $M(\mathbb{T})$.

Zafran's measures

The following result is the most important theorem from Zafran's paper ($M_0(\mathbb{T})$ is an ideal of $M(\mathbb{T})$ consisting of measures with Fourier-Stieltjes transforms vanishing at infinity).

Theorem of Zafran

Let $\mathscr{C} = M_0(\mathbb{T}) \cap \mathcal{N} = \{\mu \in M_0(\mathbb{T}) : \sigma(\mu) = \overline{\widehat{\mu}(\mathbb{Z})} = \widehat{\mu}(\mathbb{Z}) \cup \{0\}\}$. Then:

1. The set $\mathscr{C}$ is a closed ideal in $M(\mathbb{T})$.
2. For $\varphi \in \triangle(M(\mathbb{T})) \setminus \mathbb{Z}$ and $\mu \in \mathscr{C}$ we have $\varphi(\mu) = 0$.
3. $\triangle(\mathscr{C}) = \mathbb{Z}$

We are going to prove the spectral reasonability of Zafran's measures.

Zafran's measures are reasonable I

We will use one simple lemma.

Lemma

Let $\lambda \in \sigma(\mu)$ be an isolated point of $\sigma(\mu)$. Then $\lambda \in \widehat{\mu}(\mathbb{Z})$.

We are ready now to prove $\mathscr{C} \subset \mathcal{S}$.

Spectral reasonability of Zafran's measures

All Zafran's measures are spectrally reasonable, i.e. $\mathscr{C} \subset \mathcal{S}$.

Zafran's measures are reasonable II

Let us take $\mu \in \mathscr{C}$ and $\nu \in \mathscr{N}$. By the basic properties of the spectrum and by Zafran's theorem, we have

$$\sigma(\mu+\nu) = \{\varphi(\mu+\nu) : \varphi \in \triangle(M(\mathbb{T}))\} =$$
$$= (\widehat{\mu}+\widehat{\nu})(\mathbb{Z}) \cup \{\varphi(\nu) : \varphi \in \triangle(M(\mathbb{T})) \setminus \mathbb{Z}\}.$$

If $\varphi(\nu)$ for some $\varphi \in \triangle(M(\mathbb{T})) \setminus \mathbb{Z}$ is an accumulation point of $\sigma(\mu+\nu)$, then there are $\varphi_k \in \triangle(M(\mathbb{T}))$ such that $\varphi_k(\mu+\nu) \to \varphi(\nu)$ as $k \to \infty$. Without losing the generality we can assume that $\varphi_k \notin \mathbb{Z}$ for all $k \in \mathbb{N}$ (otherwise $\varphi(\nu) \in \overline{(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})}$ and we are done). Using Zafran's theorem once again, we obtain $\varphi_k(\mu+\nu) = \varphi_k(\nu)$ which implies that $\varphi(\nu)$ is an accumulation point of $\sigma(\nu) = \overline{\widehat{\nu}(\mathbb{Z})}$. Hence there is a sequence of integers $(n_k)_{k=1}^{\infty}$ such that $|n_k| \xrightarrow[k\to\infty]{} \infty$ and $\widehat{\nu}(n_k) \xrightarrow[k\to\infty]{} \varphi(\nu)$. Recalling that $\mathscr{C} \subset M_0(\mathbb{T})$ we obtain $(\widehat{\mu}+\widehat{\nu})(n_k) \xrightarrow[k\to\infty]{} \varphi(\nu)$ which finishes the proof since in case $\varphi(\nu)$ is an isolated point of $\sigma(\mu+\nu)$ we are able to use the lemma.

Discrete measures are not reasonable I

We are going to show that discrete measures are not spectrally reasonable. The starting point is the lemma on fat divisors of zero.

Lemma on fat divisors of zero

Let $\mu \in M(\mathbb{T})$ and let $r := r(\mu)$ (spectral radius). Then, for every measure $\nu \in M(\mathbb{T})$ such that

$$\overline{\widehat{\nu}(\mathbb{Z})} = r\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq r\} \text{ and } \mu * \nu = 0$$

we have $\mu + \nu \in \mathscr{N}$.

From $\mu * \nu = 0$ we have

$$\forall_{\varphi \in \triangle(M(\mathbb{T}))} \varphi(\mu) = 0 \text{ or } \varphi(\nu) = 0 \qquad (1)$$

which gives $\sigma(\mu) = \{\varphi(\mu) : \varphi \in \triangle(M(\mathbb{T}))\} = \{\varphi(\mu) : \varphi(\nu) = 0\} \cup \{0\}$, $\sigma(\nu) = \{\varphi(\nu) : \varphi(\mu) = 0\} \cup \{0\}$.

Applying consecutively (1), the last assertion and $\overline{\widehat{\nu}(\mathbb{Z})} = r\overline{\mathbb{D}}$ we obtain

$$\sigma(\mu+\nu)\cup\{0\} = \{\varphi(\mu+\nu) : \varphi\in\triangle(M(\mathbb{T}))\}\cup\{0\} =$$
$$\{\varphi(\mu):\varphi(\nu)=0\}\cup\{\varphi(\nu):\varphi(\mu)=0\}\cup\{0\} = \sigma(\mu)\cup\sigma(\nu) = r\overline{\mathbb{D}}.$$

The spectrum of an element in a unital Banach algebra is closed which gives $\sigma(\mu+\nu) = r\overline{\mathbb{D}}$. Analogously,

$$(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})\cup\{0\} = \widehat{\mu}(\mathbb{Z})\cup\widehat{\nu}(\mathbb{Z})\cup\{0\}.$$

Hence,

$$r\overline{\mathbb{D}} = \overline{\widehat{\nu}(\mathbb{Z})} \subset \overline{(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})}\cup\{0\} = \overline{(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})} \subset \sigma(\mu+\nu) = r\overline{\mathbb{D}}.$$

This precisely means $\sigma(\mu+\nu) = \overline{(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})}$ which finishes the proof.

Discrete measures are not reasonable III

We are ready to prove the following (quite surprising) theorem.

Rational Dirac deltas

Let $\alpha \in \pi\mathbb{Q} \setminus \{0\}$. Then $\delta_\alpha \notin \mathcal{S}$.

We present the proof only for the case $\alpha = \pi$. Suppose on the contrary that $\delta_\pi \in \mathcal{S}$. Then, of course, $\frac{\delta_0+\delta_\pi}{2} \in \mathcal{S}$. Consider the Riesz product

$$R = \prod_{k=1}^{\infty}(1 + \cos(4^k t)).$$

For such R we have $\mathrm{supp}\widehat{R} \subset 4\mathbb{Z} \subset 2\mathbb{Z}$. Let us define a measure μ by the formula

$$\mu = R + \frac{\delta_0 - \delta_\pi}{2} * \frac{\delta_\alpha + \delta_\beta}{2}$$

where $\alpha, \beta \in \mathbb{T}$ are such that the set $\{1, \alpha, \beta\}$ is linearly independent over $\mathbb{Q}$. Then, for $v = \frac{\delta_0-\delta_\pi}{2} * \frac{\delta_\alpha+\delta_\beta}{2}$ we have $\overline{\widehat{\nu}(\mathbb{Z})} = \overline{\mathbb{D}}$.

Discrete measures are not reasonable IV

Moreover, since $\frac{\delta_0-\delta_\pi}{2}*R=0$ we obtain from the Lemma on fat divisors of zero that $\mu\in\mathcal{N}$. One can prove $\mathcal{S}*\mathcal{N}\subset\mathcal{N}$ which gives $\frac{\delta_0+\delta_\pi}{2}*\mu\in\mathcal{N}$. But

$$\mu*\frac{\delta_0+\delta_\pi}{2}=R*\frac{\delta_0+\delta_\pi}{2}=R$$

which is a contradiction.
The theorem of non-reasonability of discrete measures in full generality (with much more involved proof) states as follows:

Discrete measures are not reasonable IV

Moreover, since $\frac{\delta_0-\delta_\pi}{2} * R = 0$ we obtain from the Lemma on fat divisors of zero that $\mu \in \mathcal{N}$. One can prove $\mathcal{S} * \mathcal{N} \subset \mathcal{N}$ which gives $\frac{\delta_0+\delta_\pi}{2} * \mu \in \mathcal{N}$. But

$$\mu * \frac{\delta_0 + \delta_\pi}{2} = R * \frac{\delta_0 + \delta_\pi}{2} = R$$

which is a contradiction.
The theorem of non-reasonability of discrete measures in full generality (with much more involved proof) states as follows:

Non-reasonability of discrete measures

If $\mu \in \mathcal{S} \cap M_d(\mathbb{T})$ then μ is a constant multiple of δ_0.

Thank you for your attention!