Geometry of Dilated Systems and Root Systems of Non-Selfadjoint operators

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Abstract

We discuss completeness, minimality, and basisness, in L^2 and L^p, $p \neq 2$, of systems of functions in three families:

(a) eigensystems, or root systems, of Hill operators on a finite interval, e.g.

$$Ly = -y'' + v(x)y, \quad 0 \le x \le \pi,$$

with periodic, antiperiodic, or Dirichlet boundary conditions;

(b) root systems of the perturbed Harmonic Oscillator Operator

$$Hu = -u'' + x^2 u + b(x)u, \quad x \in \mathbb{R}^1,$$

(c) dilated systems $u_n(x) = S(nx)$, $n \in \mathbb{N}$, where S is a trigonometric polynomial

$$S(x) = \sum_{k=0}^{m} a_k \sin(kx), \quad a_0 a_m \neq 0, \quad 0 \le x \le \pi.$$

Abstract

We will present a series of results on the systems (a), (b), (c), from [1] – [5] and more, and mention a few unsolved questions.

Introductory References I

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Introductory References II

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Outline

Hill Operator

(a) The Hill operator is

$$Ly = -y'' + v(x)y, \quad 0 \le x \le \pi,$$

with the regular b.c.

$$\text{Per}^{\pm} \quad y(\pi) = \epsilon y(0),\ y'(\pi) = \epsilon y'(0),\ \epsilon = \pm 1$$

and the strictly regular b.c.

$$\text{Dir} \quad y(\pi) = y(0) = 0.$$

The eigenvalues are squares n^2, with notations, parities, and multiplicities as below:

Eigenvalues		Multiplicity of eigenvalue at n^2 for n: Even	Odd
λ_n^+	Per^+	2	0
λ_n^-	Per^-	0	2
μ_n	Dir	1	1

Hill Operator, Cont'd

v is a complex-valued potential, $v(x+\pi) = v(x)$, $v \in L^2, \ldots, H^{-1}$

$$f = S_N f + \sum_{n>N} P_n f \tag{1}$$

$$P_n = \frac{1}{2\pi i} \int_{|z-n^2|=1/4} R(z; L_{bc})\, dz \tag{2}$$

strictly regular b.c.

1962 V.P. Mikhailov

1964 G.M. Keselman

regular b.c.

1979 A. A. Shkalikov

Counterexamples: divergence in regular (non-strictly) case

2006 Plamen Djakov –B. Mityagin,
A. Makin

$$v(x) = \sum_{n>0} \frac{e^{i2nx}}{n^\alpha} + \sum_{n<0} \frac{e^{-2inx}}{(-n)^\beta}, \quad \alpha \neq \beta, \quad \frac{1}{2} < \alpha, \beta < 1.$$

In $\mathbb{C}^2$

$$Pf = \langle f, \psi^1 \rangle \varphi^1 + \langle f, \psi^2 \rangle \varphi^2, \quad \langle \psi^j, \varphi^k \rangle = \delta_{jk}$$

$$|\langle \bullet, \psi^1 \rangle \varphi^1| = (\sin \alpha)^{-1}$$

$$(\sin \alpha)^2 = 1 - \frac{|\langle \psi^1, \psi^2 \rangle|^2}{\langle \psi^1, \psi^1 \rangle \cdot \langle \psi^2, \psi^2 \rangle}$$

Main Claim

Claim

(P. Dj. – B. Mi.)

(i) $v = ae^{-2ix} + be^{2ix}$

(ii) $v = ae^{-2ix} + Be^{4ix}$

(iii) $v = ae^{-2ix} + Ae^{-4ix} + be^{2ix} + Be^{4ix}$

Convergence? Yes:

(i) IFF $|a| = |b|$

(ii) NO, for Per$^+$

(iii) YES, iff $|A| = |B|$.
(but we exclude the case when $-\frac{b^2}{4B}, -\frac{a^2}{4A}$ are exact squares m^2, $m \in \mathbb{N}$).

Technical Tools B^+, B^-

Definition

For $n \in \mathbb{N}$ a walk from $-n$ to $+n$ (or from $+n$ to $-n$ is defined as a sequence of steps $x = \{x(t)\}_{t=1}^{\nu+1}$, $1 \le \nu = \nu(x) < \infty$ where $x(t) \in 2\mathbb{Z} \setminus \{0\}$, and $\sum_{t=1}^{\nu+1} x(t) = 2n$ [or $-2n$].
A walk is admissible if its vertices

$$j(t) = j(t, x) = -n + \sum_{i=1}^{t} x(i), \quad [\text{or } +n + \sum_{i=1}^{t} x(i)],$$

$$1 \le t \le \nu + 1, \quad j(0) = -n \; [\text{or } +n]$$

satisfy $j(t) \ne \pm n$, $1 \le t \le \nu$.

Technical Tools II

Let $X_n^{\pm}$ be the set of all admissible walks from $-n$ to $-n$ (respectively, $+n$ to $-n$). For each walk $x \in X_n^{\pm}$ set

$$h(x, \zeta, \nu) = \frac{\prod_{t=1}^{\nu+1} \nu(x(t))}{\prod_{t=1}^{\nu} \left[n^2 - j(t)^2 + \zeta\right]}$$

and define

$$B^{\pm}(n, \zeta) = \sum_{x \in X_n^{\pm}} h(x, \zeta, \nu).$$

Technical Tools III

(i)

$$B^{+}(n,\zeta) = h(\omega^*,0)\left(1 + O\left(\frac{\log n}{n}\right)\right)$$

$$\omega^*(t) = 2$$

$$h(\omega^*,0) = 4\left(\frac{b}{4}\right)^n [(n-1)!]^2$$

$$B^{-}(n,\zeta) = 4\left(\frac{a}{4}\right)^n [(n-1)!]^2$$

Technical Tools IV

(iv) $ae^{-2irx} + be^{+2isx}$, $a, b \neq 0$; $r, s \in \mathbb{N}$, $r \neq s$.
NO if:

- $\mathrm{bc} = \mathrm{Per}^{+}$,
- or $\mathrm{bc} = \mathrm{Per}^{-}$, r, s odd
- or $\mathrm{bc} = \mathrm{Per}^{-}$, $r = 1$, $s \geq 3$.

Que. $r = 1$, $s = 2$, Per^{-}.
"NO" but our difficulties are combinatorial.
An obstacle: the Catalan Numbers identity

$$C_{k+1} = \sum_{i=1}^{k} C_i C_{k+1-i}$$

where

$$C_k = \frac{1}{k}\binom{2k-2}{k-1}, \quad k \geq 1.$$

1 Hill Operator

2 Dilated Systems
- Single-Frequency Case
- Multi-Frequency Case

3 Root Systems of Harmonic Oscillator Operator

1. Hill Operator

2. Dilated Systems
   - Single-Frequency Case
   - Multi-Frequency Case

3. Root Systems of Harmonic Oscillator Operator

Dilated Systems of Trigonometric Polynomials

Let $u_n(x) = S(nx)$, $n \geq 1$, where

$$S(x) = \sum_{j=0}^{m} a_j \sin(2^j x),$$

analyzed in $L^p[0, \pi]$, $1 < p < \infty$.

1936 L.A. Lyusternik

1970 J. Neuwirth – J. Ginsberg – D. Newman

Define the polynomial

$$a(z) = \sum_{j=0}^{m} a_j z^j,$$

where

$$\begin{array}{ccccccc} Z(a) & = & F^- & \cup & F^0 & \cup & F^+ \\ a(\alpha) = 0 & & |\alpha| < 1 & & |\alpha| = 1 & & |\alpha| > 1. \end{array}$$

Dilated Systems II

Note: the isometry

$$T : f(x) \to f(2x),$$

has spectrum

$$\sigma(T) = \overline{\mathcal{D}}, \quad \mathcal{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}.$$

Then

$$u_n = a(T)\{\sin(nx)\}.$$

We factorize

$$a(z) = a^{-}(z)a^{0}(z)a^{+}(z).$$

Zeroes outside the unit circle present no obstacle

We note that $a^+(T)$ is invertible, for

$$B = \prod_{\alpha \in F^+} (\alpha - T)^{-1}$$

$$= \frac{1}{2\pi i} \int_{|z|=1+\delta} R(z,t) \frac{dz}{a^+(z)} ,$$

$$1 + 2\delta = \min\{|\alpha| : \alpha \in F^+\}.$$

Let $v_n = Bu_n = a^-(T)a^0(T)\{\sin(nx)\}$.

Claim 1

$U = \{u_n\}$ is a basis in $L^2[0,\pi]$ if and only if $F^- \cup F^0 = \emptyset$.

Zeroes inside the unit circle I

Claim 2

If $F^- \neq \emptyset$, the system U is not complete. [I. Schur, R. Cohen]

Proof

Indeed, put for $a(\alpha) = 0$, $|\alpha| < 1$,

$$h(x) = \sum_{k=0}^{\infty} \alpha^k \sin(2^k x).$$

Letting $\{w \text{ odd }\} = \Omega$, define

$$\mathbb{N}(\omega) = \left\{\omega \cdot 2^k : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right\}$$

so

$$\mathbb{N} = \bigcup_{\omega \in \Omega} \mathbb{N}(\omega), \quad \mathbb{N}(\omega) \cap \mathbb{N}(\omega') = \emptyset \text{ if } \omega \neq \omega'.$$

Zeroes inside the unit circle II

Proof.

Then for $n = \omega \cdot 2^{k_0}$, $\omega \neq 1$,

$$\langle h, u_n \rangle = \int_0^\pi h(x) u_n(x)\, dx = 0,$$

(no bar, no conjugation) and for $\omega = 1$,

$$\langle h, u_n \rangle = \sum_{k=0}^{m} a_k \langle h, \sin(2^{k_0+k} x) \rangle =$$

$$= \frac{\pi}{2} \sum_{k=0}^{m} a_k \alpha^{k_0+k} = \frac{\pi}{2} \alpha^{k_0} a(\alpha) = 0.$$

So $H(f) = \int_0^\pi h(x) f(x)\, dx$ is a non-zero bounded linear functional on L^p, $1 \leq p < \infty$, such that $H(u_n) = 0$, for all n. $\square$

Zeroes on Unit Circle I

So assume $(\star)$ $F = F^0 \neq \emptyset$, i.e., all roots α, $a(\alpha) = 0$, are in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Claim 3

Under $(\star)$

$$a(z) = a_m \prod_{\alpha \in F^0} (\alpha - z)^{\mu(\alpha)}.$$

Put $\kappa^* = \max\{\mu(\alpha) - 1 : \alpha \in F^0\}$. The system U is complete, and minimal, i.e., $\exists \{\Phi_k\}$, $\langle \Phi_k, u_n \rangle = \delta_{kn}$, and

$$\|\Phi_k\|_q \asymp (\log k)^{\kappa^* + 1/2},$$

so U is not a basis in L^2 (or L^p, $1 < p < \infty$).

Zeroes on Unit Circle II

How to find $\{\Phi_k\}$? Let

$$n = \omega 2^{k_0}, \quad \Phi_n \sim \varphi(x) \in L^q$$

$$\varphi(x) = Q(\omega)\varphi = \sum_{k=0}^{\infty} Y_k \sin(\omega 2^k x), \quad j \in \mathbb{N}(\omega), \quad j = \omega 2^k;$$

then

$$\frac{2}{\pi}\int_0^\pi \varphi(x) u_j(x)\, dx = \sum_{i=0}^{m} a_i Y_{k+i}$$

and we are looking for a solution of a non-homogeneous infinite system

$$\sum_{i=0}^{m} a_i Y_{k+i} = \begin{cases} 0, & 0 \le k < k_0; \\ 1, & k = k_0; \\ 0, & k > k_0. \end{cases} \tag{$\dagger$}$$

Zeroes on Unit Circle III

If $\{X_j, j \in \mathbb{Z}\}$ solves such a system for $k_0 = 0$ and $k < 0$ then for any $k_0 \geq 0$

$$Y_k = X_{k-k_0}$$

solves (†)

$$X_k = 0, k > 0; \qquad X_0 = \frac{1}{a(0)}$$

$$X_{-k} = \frac{1}{k!}\left(\frac{d}{dt}\right)^k \left[\frac{1}{a(t)}\right]\Bigg|_{t=0}, \quad k \geq 0.$$

Claim

The system U is minimal for any polynomial $a(z)$, $a_0 \neq 0$.

(Inequalities for the norms of the inverses of Vandermonde matrices)

1. Hill Operator

2. Dilated Systems
   - Single-Frequency Case
   - Multi-Frequency Case

3. Root Systems of Harmonic Oscillator Operator

We again set $U = \{u_n(x)\}$ and $u_n(x) = S(nx)$, but now

$$S(x) = \sum_{j \in J} a_j \exp(ijx), \ |J| < \infty$$

$$= \sum_{\substack{\alpha \in \mathbb{N}_0^m \\ \alpha \in K,\ K \subseteq \mathbb{N}_0^m}} a(\alpha) \exp\left(i \left[\prod_{j=1}^m p_j^{\alpha_j} \right] x \right), \ |K| < \infty.$$

is modeled by a multi-variable polynomial

$$A(\omega) = \sum_{\alpha \in K} a(\alpha) w^\alpha, \quad w^\alpha = \prod_{j=1}^m w_j^{\alpha_j}.$$

Multifrequency Case II

We use the isometries $T_j : f(x) \mapsto f(p_j x),\ 1 \leq j \leq m$. We adjust the sets

$$\begin{aligned} \mathbb{N}(\omega) &= \{\omega \cdot p^\alpha : \alpha \in \mathbb{N}_0^m\} \\ \omega \in \Omega &= \{q \in \mathbb{N} : q \text{ does not have factors } p_j,\ 1 \leq j \leq m\}. \end{aligned}$$

Consider

$$E \cong \ell^2 \simeq H^2(\mathcal{D}) \simeq \ell^2 \left(\Omega; H^2(\mathcal{D}^m)\right).$$

All $E(\omega) = \operatorname{Im} Q(\omega)$, $\omega \in \Omega$, are invariant with respect to T_j, $1 \leq j \leq m$, i.e. multiplication by w_j in $H^2(\mathcal{D}^m)$. Certainly,

$$\|f\|^2 = \sum_{\omega \in \Omega} \|Q(\omega) f\|^2.$$

Easy Claims

Now all the questions about U become the questions about the system

$$\begin{aligned} V &= \{v(\alpha)\}_{\alpha \in \mathbb{N}_0^m}, \\ &= v(\alpha)(w) = A(w) w^{\alpha}, \end{aligned} \qquad \text{in } H^2(\mathcal{D}^m).$$

Claim

If

$$Z(A) \cap \overline{\mathcal{D}^m} = \emptyset, \qquad (\star\star)$$

then $A(T)^{-1} = B$ is well-defined and $\{v(\alpha)\}$ is a (Riesz) basis and U is a (Riesz) basis as well.

If V (or U) is a Riesz basis then $(\star\star)$ holds.

Minimality

Claim

The system U is minimal if $a(0) \neq 0$.

Indeed, with

$$\langle f, g \rangle = \frac{1}{(2\pi)^m} \int\limits_{\mathbb{T}^m} f(w) g(w) \, \mathrm{d}^m t,$$

$$w_j = e^{it_j}, \quad 1 \leq j \leq m,$$

(no bar, no conjugation),

$$\langle w^{-\tau}, w^{\alpha} \rangle = \delta(\alpha, \tau), \quad \forall \alpha, \tau \in \mathbb{Z}^m.$$

Minimality II

Try an adjustment

$$\langle \frac{1}{A(w)} w^{-\tau}, A(w) w^{\alpha} \rangle \overset{??}{=} \delta(\alpha, \tau)$$

$$\frac{1}{A(w)} = \sum_{\sigma \in \mathbb{N}_0^m} b(\sigma) w^{\sigma}$$

$$\frac{w^{-\tau}}{A(w)} = \sum_{\sigma \in \mathbb{N}_0^m} b(\sigma) w^{\sigma - \tau}$$

but if $\sigma - \tau \leq 0$ does not hold then

$$\langle w^{\sigma-\tau}, w^{\alpha} \rangle = 0, \quad \forall \alpha \in \mathbb{N}_0^m$$

so put

$$\Phi_t(w) = \sum_{\sigma \leq \tau} b(\sigma) w^{\sigma - \tau}.$$

Minimality III

$$\Phi_t(w) = \sum_{\sigma \le \tau} b(\sigma) w^{\sigma - \tau}.$$

This finite sum is well-defined, and

$$\langle \Phi_\tau, v(\alpha) \rangle = \delta(\alpha, \tau), \qquad \text{for all } \alpha, \tau \in \mathbb{N}_0^m.$$

Completeness I

$$Z(A) \cap \overline{\mathcal{D}^m} = Z(A) \cap \mathbb{T}^m.$$

1970 NGN

$$A(w)/A(rw) \to 1 \quad \text{a.e. on } \mathbb{T}^m.$$

$$|A(w)/A(rw)| \leq 2^{\deg A}$$

Completeness implies uniquenss of the system Φ_τ and the fact that for 1D projectors $P_\tau = \langle \bullet, \Phi_\tau \rangle v(\tau)$,

$$\|P_\tau\| = \|\Phi_\tau\| \cdot \|v(\tau)\| \asymp \|\Phi_\tau\|.$$

Completeness II

But

$$\|\Phi_\tau\|^2 = \sum_{\sigma \le \tau} |b(\sigma)|^2, \quad B(w) = \frac{1}{A(w)},$$

so these norms are uniformly bounded if and only if

$$\frac{1}{A(w)} \in H^2(\mathcal{D}^m), \quad \text{or}$$

$$\frac{1}{P(t)} \in L^1(\mathbb{T}^m),$$

$$P(t) = |A(e^{it})|^2.$$

Incompleteness with few frequencies

Claim

If $m \leq 3$ and $Z(A) \cap \overline{\mathcal{D}^m} \neq \emptyset$, then

$$\frac{1}{A(w)} \notin H^2(\mathcal{D}^m).$$

Corollary

Under the same assumptions, V or U is NOT a basis.

Obstruction to Incompleteness Proof with many frequencies

$m \geq 4$. It could happen that $\dfrac{1}{A(w)} \in H^2(\mathcal{D}^m)$.

Example

Fix $c_k > 0$, $1 \leq k \leq m$, with $\sum_{k=1}^m c_k = 1$, and define

$$A(w) = 1 - \sum_{k=1}^{m} c_k w_k. \tag{E^*}$$

On $\mathbb{T}^m$ this equals

$$\sum_{k=1}^{m} c_k(1 - e^{it_k}) = \sum_{k=1}^{m} c_k \left[(1 - \cos t_k) - i \sin t_k\right].$$

Obstruction to Incompleteness II

Then

$$P(t) = |\sum_{k=1}^{m} c_k 2\sin^2\left(\frac{t_k}{2}\right)|^2 + |\sum_{k=1}^{m} c_k \sin(t_k)|^2$$

$$\asymp r^4 + |\ell(t)|^2, \quad \ell(t) = \sum_{k=1}^{m} c_k t_k, \quad |t| \ll 1.$$

We note that $\displaystyle\int_{|\zeta|\leq\delta} \frac{d\zeta_0\, d\zeta_1 \ldots d\zeta_{m-1}}{\zeta_0^2 + \zeta_1^4 + \cdots + \zeta_{m-1}^4} < \infty$ if and only if $m \geq 4$, since it is within a constant multiple of

$$\int_0^\delta \int_0^{\rho^2} \frac{d\zeta\, \rho^{m-2}\, d\rho}{\zeta^2 + \rho^4} = \int_0^\delta \int_0^1 \frac{d\eta\, \rho^{m-4}\, d\rho}{1 + \eta^2}.$$

Recall that $v(\alpha) = A(w)w^\alpha$, $\alpha \in \mathbb{N}_0^m$. Instead of asking whether this system is a basis in $H^2(\mathcal{D}^m)$, or $\ell^2(\mathbb{N}_0^m)$ we can move to weighted $H^2(\mathcal{D}^m; P)$, or $L^2(\mathbb{T}^m; P)$ and ask whether $\{w^\alpha\}$ is a basis.

Claim

In the case (E^*), $m \geq 4$, for the partial sums

$$\Sigma(\tau)f = \sum_{\alpha \leq \tau} \langle \Phi_\alpha, f \rangle v_\alpha,$$

$\|\Sigma(\tau)\|$ are not bounded.

The proof comes from the multi-dimensional A_2 Muckenhoupt condition.

1988 Kazarian – Lizorkin

2010 Kabe Moen

The weight $P(t) = \ell(t)^2 + \rho^4$ is not good. (!!!)

But if we go to the original system

$$v_\alpha(x) = \sum_{k=0}^{K} a_k \exp\left(i[p^\alpha]kx\right), \quad p = (p_j)_{j=1}^m, \quad \alpha \in \mathbb{N}_0^m,$$

its linear ordering would fit to a monotone arrangement of the multi-index sequence $\{p^\alpha\}$, or its linear ordering by monotonicity of the linear form

$$M(\alpha) = \sum_{j=1}^{m} \alpha_j \log p_j.$$

It leads us to the question on the boundedness of the projection Q_M.

$$\exp i(\alpha, t) \to \text{ the same, if } M(\alpha) \geq 0$$
$$\to 0 \text{ if} M(\alpha) < 0.$$

in the weighted $L^2(\mathbb{T}^m; P)$, $m \geq 4$, say,

$$P(t) = \left(\sum_{j=1}^{m} t_j \right)^2 + \left(\sum_{j=1}^{m} t_j^2 \right)^2 .$$

$M(\alpha)$ is "essentially irrational," i.e., the coefficients

$$\mu_j^* = \log p_j, \quad 1 \leq j \leq m,$$

of the linear function

$$M(y) = \sum_{j=1}^{m} \mu_j y_j$$

are rationally independent.

If all μ_j were rational Q_M would be equivalent to the case $M_0(y) = y_1$; then known A_2 conditions are applicable and Q_{M_0} is unbounded, for any m.
If Q_{M^*} were bounded, $m \geq 4$ (I do not believe so) we would have Babenko–type Shauder (but not Riesz) basis in $H^2(\mathcal{D}^m)$, or even in $L^2(\mathbb{T}^m)$.
1948 $L^2[-\pi, \pi]$

$$\text{Take } A(t) = \begin{cases} |t|^\alpha, & -\pi \leq t \leq 0; \\ |t|^\beta, & 0 \leq t \leq \pi, \end{cases}$$

for $0 < \alpha, \beta < \frac{1}{2}$. $v_n(t) = A(t)e^{int}$, $n \in \mathbb{Z}$ is a (conditional) basis if $\alpha = \beta$, but is NOT a basis if $\alpha \neq \beta$.

Harmonic Oscillator Perturbation I

(b) Let

$$T = -\frac{d^2}{dx^2} + x^2,$$
$$\mathfrak{D}(T) = \{f \in W^{2,2}(\mathbb{R}) : x^2 f \in L^2(\mathbb{R})\}$$

and

$$B = 2iax, \quad a \in \mathbb{R} \setminus \{0\},$$
$$\mathfrak{D}(B) = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}$$

We consider the operator $L = T + B$.

Eigenfunctions of unperturbed operator

The eigenvalues and eigenfunctions of T are well-known; we have

$$Th_k = (2k+1)h_k, \quad k \geq 0,$$

where h_k are the Hermite functions given by

$$h_k = (2^k k! \sqrt{\pi})^{-1/2} H_k(x) e^{-x^2/2},$$

where $H_k(x)$ are the Hermite polynomials given by

$$H_k = e^{x^2/2} \left(x - \frac{d}{dx} \right)^k e^{-x^2/2}.$$

Hermite Function Asymptotics

We have the following asymptotic for the Hermite functions,

$$h_m(x) = \frac{2^{1/4}}{\pi^{1/2}} \frac{1}{m^{1/4}} \left[\cos\left(x\sqrt{2m+1} - m\frac{\pi}{2} \right) \right. \\ \left. + \frac{x^3}{6} \frac{1}{\sqrt{2m+1}} \sin\left(x\sqrt{2m+1} - m\frac{\pi}{2} \right) + O\left(\frac{1}{m} \right) \right].$$

Hermite Function L^p norm Inequalities I

Lemma

As $n \to \infty$,

$$\|h_n\|_r \sim n^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)}, \quad 1 \le r < 4$$
$$\|h_n\|_r \sim n^{-\frac{1}{8}} \log n, \quad r = 4$$
$$\|h_n\|_r \sim n^{-\frac{1}{6}\left(\frac{1}{r}+\frac{1}{2}\right)}, \quad r > 4$$

See 1993 Thangavelu [Lemma 1.5.2] for the sketch of the proof and further explanations of these claims.

Hermite Function Inequalities II

If $p > 2$, and q is the conjugate exponent to p, then $q < 2$, $2q < 4$ so

$$\|h_k\|_{2q} \sim k^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2q}\right)} = k^{-\frac{1}{4p}}, \qquad p > 2.$$

For $p = 2$ we have $2q = 4$ and

$$\|h_k\|_4 \sim k^{-\frac{1}{8}} \log k, \qquad p = 2.$$

Finally, if $1 \leq p < 2$ then $2q > 4$ so

$$\|h_k\|_{2q} \sim k^{-\frac{1}{6}\left(\frac{1}{2q}+\frac{1}{2}\right)} = k^{-\frac{1}{12}\left(2-\frac{1}{p}\right)}, \qquad 1 \leq p < 2$$

Biorthogonal expansion I

Let

$$
\begin{aligned}
f_k &= h_k(x+ia), \\
g_k &= h_k(x-ia),
\end{aligned}
$$

so

$$
\begin{aligned}
Lf_k &= (2k+1+a^2)f_k, \\
L^* g_k &= (2k+1-a^2)g_k,
\end{aligned}
$$

and

$$
\|f_k\| = \|g_k\| = \|e^{ax} h_k(x)\|.
$$

Biorthogonal expansion II

Claim

If $P_k u = \langle u, g_k \rangle f_k$, then

$$\|P_k\| = \frac{1}{2(2k)^{1/4}\sqrt{|a|\pi}} \exp\left(2^{3/2}|a|\sqrt{k}\right)\left(1 + O\left(k^{-1/2}\right)\right), \quad k \to \infty, \tag{$*$}$$

so

$$\lim_{k\to\infty} \frac{1}{\sqrt{k}} \log\|P_k\| = 2^{3/2}|a|.$$

Claim

Let $B \sim p(x)\dfrac{d}{dx} + q(x)$. For any σ, $0 < \sigma \leq 1$, we can choose B such that

$$\lim_{k\to\infty} k^{-\sigma}\|P_k\| = c, \quad 0 < c < \infty.$$

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Question: What about $-\dfrac{d^2}{dx^2} + q(x)$?

Number of Nonreal Eigenvalues

We consider the cases $b(x) \in L^p(\mathbb{R})$, $1 \le p < \infty$, or

$$b(x) = \sum_k c_k \delta(x - \tau_k).$$

Claim

If

(i) $\overline{b}(x) = -b(-x)$, and

(ii) $\|b|L^1\| = \sigma$, or $\sum_k |c_k| = \sigma > 2$,

then the number of non-real eigenvalues

$$N^* \le A(\sigma \log \sigma)^6.$$

If also (iii) supp b is bounded, then

$$N^* \le A(\sigma \log \sigma)^2.$$ 2014–15 B. Mityagin

Thank You

The End

Ending frame for preserving TOC