On bounded perturbations of Bernstein functions of several semigroup generators on Banach spaces

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The study of the problem of differentiability of functions of self-adjoint operators on Hilbert space was initiated By Yu. Daletskiĭ and S.G. Kreĭn. Much work has been done during last decades on the theory of classes of operator Lipschitz and Frechet-differentiable functions of operators on Hilbert space by Birman and Solomyak, Aleksandrov, Peller, Nazarov, Farforovskaya, Nikolskaya, Kissin, Potapov, Shulman, Sukochev, Naboko, Davies, Johnson and Williams, Arazy, Barton, Friedman, Pedersen, and others.

Bernstein functions in n variables

Definition (Bochner). A function $\psi \in C^{\infty}((-\infty; 0)^n)$, $\psi \le 0$ is a *nonpositive Bernstein function* in n variables ($\psi \in \mathcal{BF}_n$) if $\forall \frac{\partial \psi}{\partial s_j}$ are absolutely monotone ($\Leftrightarrow -\psi(-s)$ is a nonnegative Bernstein function).

Function $\psi \in \mathcal{BF}_n$ admits an integral representation

$$\psi(s) = c_0 + c_1 \cdot s + \int\limits_{\mathbb{R}^n_+ \setminus \{0\}} (e^{s \cdot u} - 1) d\mu(u) \quad (s \in (-\infty; 0)^n), \tag{1}$$

Examples ($n = 1$): $e^s - 1$; $-(-s)^{\alpha}$, $\alpha \in [0, 1]$; $-\log(1 - s)$.

Multidimensional Bochner-Phillips functional calculus

In the following $T_{A_1}, \ldots, T_{A_n}$ denote pairwise commuting one-parameter C_0 semigroups on a complex Banach space X with generators $A_1, \ldots, A_n$ respectively such that $\|T_{A_j}(t)\| \le M$ (we write $A = (A_1, \ldots, A_n) \in \mathrm{Gen}(X)^n$).

Definition [a&A, 1999]. The value of a function $\psi \in \mathcal{BF}_n$ of the form (1) at $A \in \mathrm{Gen}(X)^n$ applied to $x \in \cap_j D(A_j)$ is defined by

$$\psi(A)x := c_0 x + c_1 \cdot Ax + \int\limits_{\mathbb{R}^n_+ \setminus \{0\}} \left(\prod_{j=1}^{n} T_{A_j}(u_j) - I \right) x d\mu(u), \qquad (2)$$

$$\psi(A) := \mathrm{closure}(\psi(A)|D(A); \ \psi : \mathrm{Gen}(X)^n \to \mathrm{Gen}(X).$$

Bounded perturbations of Bernstein functions

Theorem 1 *Let $\psi \in \mathcal{BF}_n$. For every commuting families $A = (A_1, \ldots, A_n)$, and $B = (B_1, \ldots, B_n)$ from* $\mathrm{Gen}(X)^n$ *such that all $A_j - B_j \in \mathcal{L}(X)$ we have $\psi(A) - \psi(B) \in \mathcal{L}(X)$ and the following inequality holds*

$$\|\psi(A) - \psi(B)\| \leq -\frac{2e}{e-1} n M^n \psi\left(-\frac{M}{2n}\|A - B\|\right), \tag{3}$$

where $\|A - B\| := (\|A_1 - B_1\|, \ldots, \|A_n - B_n\|)$.

Corollary 1 *If, in addition, all $A_j \in \mathcal{L}(X)$ then $\psi(A) \in \mathcal{L}(X)$ and the singular numbers of $\psi(A)$ satisfy the inequality*

$$s_k(\psi(A)) \leq -\frac{2e}{e-1} n M^n \psi\left(-\frac{M}{2n} s_k(A)\right) \tag{4}$$

where $s_k(A) := (s_k(A_1), \ldots, s_k(A_n)), k \in \mathbb{N}$.

$\mathcal{I}$-Lipschitzness of Bernstein functions

Below we shall assume that a (two sided) operator ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ on X is *symmetrically normed* in the sense that $\|ASB\|_{\mathcal{I}} \le \|A\|\|S\|_{\mathcal{I}}\|B\|$ for $A, B \in \mathcal{L}(X)$ and $S \in \mathcal{I}$ ($\mathcal{I}$ may be equal to $\mathcal{L}(X)$).

Theorem 2 *Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be an operator ideal on X and $\psi \in \mathcal{BF}_n$. For every commuting families $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ from* $\mathrm{Gen}(X)^n$ *such that $(\forall j)$ $A_j - B_j \in \mathcal{I}$, and $\|T_{A_j}(t)\|, \|T_{B_j}(t)\| \le Me^{\omega_j t}, \omega_j \le 0$, we have $\psi(A) - \psi(B) \in \mathcal{I}$, and*

$$\|\psi(A) - \psi(B)\|_{\mathcal{I}} \le M^{n+1} \sum_{j=1}^{n} \frac{\partial \psi(\omega_j \mathbf{e}_j)}{\partial s_j} \|A_j - B_j\|_{\mathcal{I}}. \tag{5}$$

Estimates for the norm of commutators

Theorem 3 *Let* $H \in \mathcal{L}(X)$ *be Hermitian,* $[A_j, H] \in \mathcal{I}$*, and* $\forall t \in \mathbb{R}, j = 1, \ldots, n$ $\exp(itH) : D(A_j) \to D(A_j)$*. Then* $\forall \psi \in \mathcal{BF}_n$ *such that* $(\forall j)\partial\psi(-0)/\partial s_j \neq \infty$ *and* $\psi(A) \in \mathcal{I}$*, the following inequality holds*

$$\|[\psi(A), H]\|_{\mathcal{I}} \leq M^{n+1} \sum_{j=1}^{n} \frac{\partial\psi(-0)}{\partial s_j} \|[A_j, H]\|_{\mathcal{I}}. \tag{6}$$

Frechet Differentiability $(n = 1)$

Definition 1 *Let* $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ *be an operator ideal on* X*,* $\psi \in \mathcal{BF}_1$*,* $A \in \mathrm{Gen}(X)$*. The bounded linear operator* ψ_A^{∇} *on* $\mathcal{I}$ *is the* $\mathcal{I}$*-Frechet derivative of the operator function* ψ *at the point* A*, if* $\forall \Delta A \in \mathcal{I}$ *such that* $A + \Delta A \in \mathrm{Gen}(X)$ *we have*

$$\|\psi(A + \Delta A) - \psi(A) - \psi_A^{\nabla}(\Delta A)\|_{\mathcal{I}} = o(\|\Delta A\|_{\mathcal{I}}) \text{ as } \|\Delta A\|_{\mathcal{I}} \to 0. \tag{7}$$

Theorem 4 *Let* $\psi \in \mathcal{BF}_1$*,* $\psi'(-0) \neq \infty$*,* $A \in \mathrm{Gen}(X)$*. The* $\mathcal{L}(X)$*-Frechet derivative for the operator function* ψ *at the point* A *exists and equals to* $\psi'(A)$ *in a sense that* $\forall B \in \mathcal{L}(X)$

$$\psi_A^{\nabla}(B) = \psi'(A)B. \tag{8}$$

Theorem 5 *Let* $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ *be an operator ideal on* X*,* $\psi \in \mathcal{BF}_1$*, and* $\psi'(-0) \neq \infty, \psi''(-0) \neq \infty$*. For every* $A \in \mathrm{Gen}(X)$ *the* $\mathcal{I}$*-Frechet derivative for the operator function* ψ *at the point* A *exists and equals to* $\psi'(A)$ *in a sense that* $\forall B \in \mathcal{I}$ $\psi_A^{\nabla}(B) = \psi'(A)B$*.*

Livschits-Kreĭn trace formula $(n = 1)$

Theorem 6 *Let the Banach space X has the approximation property. Let T_A and T_B are holomorphic and uniformly bounded in $\{\mathrm{Re}(z) > 0\}$. If $A - B \in \mathbf{S}_1$ then $\exists!$ distribution $\xi \in \mathcal{S}'_{\mathbb{R}_+}$ such that $\forall \psi \in \mathcal{BF}_1$ with $\psi'(-0) \neq \infty$ we have*

$$\mathrm{trace}(\psi(A) - \psi(B)) = \int\limits_{(0,\infty)} \langle \xi(t), e^{-ut} \rangle u d\mu(u), \tag{9}$$

where μ stands for the representing measure of ψ.

Corollary 2 *If, in addition, $\psi'(-t) \in \mathcal{S}_{\mathbb{R}_+}$, then*

$$\mathrm{trace}(\psi(A) - \psi(B)) = \langle \xi(t), \psi'(-t) \rangle. \tag{10}$$

Bibliography on multidimensional Bochner-Phillips functional calculus

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