Sobolev inequalities in arbitrary domains

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Let:

- Ω be an open set in $\mathbb{R}^n$, $n \geq 2$,
- $m \in \mathbb{N}$.

m-th order Sobolev inequality in Ω: bound for a norm of the h-th order weak derivatives $(0 \leq h \leq m-1)$ of any m times weakly differentiable function in Ω in terms of norms of some of its derivatives up to the order m.

Standard version. Assume that Ω is regular, e.g. a bounded Lipschitz domain.
If $p < \frac{n}{m}$, then $\exists\, C = C(\Omega, p, m)$ s.t.

$$\|u\|_{L^{\frac{np}{n-mp}}(\Omega)} \le C \sum_{k=0}^{m} \|\nabla^k u\|_{L^p(\Omega)}$$

$\forall u \in W^{m,p}(\Omega)$. Here, $\frac{np}{n-mp}$ is the critical (largest) Sobolev exponent.
If $p > \frac{n}{m}$, then $\exists\, C = C(\Omega, p, m)$ s.t.

$$\|u\|_{L^\infty(\Omega)} \le C \sum_{k=0}^{m} \|\nabla^k u\|_{L^p(\Omega)}$$

$\forall u \in W^{m,p}(\Omega)$.

More generally, if $p < \frac{n}{m}$, then

$$\|u\|_{L^{\frac{np}{n-mp}}(\Omega)} \leq C\big(\|\nabla^m u\|_{L^p(\Omega)} + \mathcal{F}(u)\big),$$

where $\mathcal{F}(\cdot)$ is any continuous seminorm in $W^{m,p}(\Omega)$ which does not vanish on any polynomial of degree not exceeding $m-1$.

Regularity of Ω is crucial in these results.

In particular, C depends on Ω.

Sobolev embeddings are spoiled in domains with bad boundaries.

Inequalities of the form mentioned above do not hold, at least with the same critical exponent $\frac{np}{n-mp}$, in irregular domains.

Suitable versions are known, with exponents depending on the geometry of the domain (for instance, domains with outward cusps).

In general, Sobolev inequalities can be shown to be equivalent to isoperimetric or isocapacitary inequalities relative to the domain [M., 1960].

New perspective: replace regularity of $\partial\Omega$ with regularity of traces of functions and their derivatives on $\partial\Omega$.
Distinctive features of our Sobolev inequalities:

- No a priori regularity on Ω. The constants are independent of the geometry of Ω.
- The critical Sobolev exponents (more generally, the optimal target norms) are the same as in the case of regular domains.
- The order of the derivatives of trial functions to be prescribed on $\partial\Omega$ is minimal.

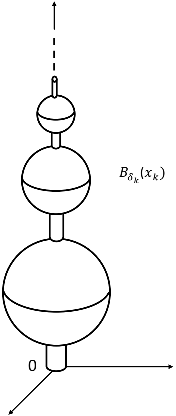
The relevant inequalities have the form:

$$\|\nabla^h u\|_{Y(\Omega,\mu)} \leq C\big(\|\nabla^m u\|_{X(\Omega)} + \mathcal{N}_{\partial\Omega}(u)\big). \tag{1}$$

Here:

- $m \in \mathbb{N}$, $h \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
- $\|\cdot\|_{X(\Omega)}$ is a Banach function norm on Ω with respect to Lebesgue measure $\mathcal{L}^n$;
- $\|\cdot\|_{Y(\Omega,\mu)}$ is a Banach function norm with respect to a possibly more general measure μ;
- $\mathcal{N}_{\partial\Omega}(\cdot)$ is a (non-standard) seminorm on $\partial\Omega$, depending on the trace of u, and of its derivatives up to the order $\left[\frac{m-1}{2}\right]$, where $[\cdot]$ stands for integer part.

The value $\left[\frac{m-1}{2}\right]$ is minimal for the bounds in question to hold without any additional assumption on Ω.
Examples of domains demonstrating this minimality are



A first-order Sobolev inequality on arbitrary open sets $\Omega \subset \mathbb{R}^n$, involving a norm on $\partial\Omega$, was proved in [M., 1960] via isoperimetric inequalities.
If $p \in [1, n)$ and $r \geq 1$, then $\exists\, C_1, C_2$ s.t.

$$\|u\|_{L^q(\Omega)} \leq C_1 \|\nabla u\|_{L^p(\Omega)} + C_2 \|u\|_{L^r(\partial\Omega)},$$

where

$$q = \min\{\tfrac{rn}{n-1}, \tfrac{np}{n-p}\}.$$

The norm $L^r(\partial\Omega)$ is taken with respect to $\mathcal{H}^{n-1}$, the $(n-1)$-dimensional Hausdorff measure.
The constants C_1 and C_2 are independent of Ω in the borderline situation when $r = \frac{p(n-1)}{n-p}$ and $q = \frac{np}{n-p}$, and just depend on $|\Omega|$ otherwise.
The optimal value of C_1 and C_2 was found in [M., 1960] for $p = 1$, and in [Maggi-Villani, 2005] for $1 < p < n$, via mass transportation techniques.

We establish arbitrary-order inequalities.
A different approach is developed, based on new representation formulas.
It yields new results even in the first-order case.

Example 1. Let Ω be any open set in $\mathbb{R}^n$, and let μ be a Borel measure on Ω such that

$$\mu(B_r \cap \Omega) \le C r^{\alpha}$$

for some $C > 0$, and $\alpha \in (n-1, n]$, and for every ball B_r radius r.
If $\mu = \mathcal{L}^n$, then $\alpha = n$.
Assume that $1 < p < n$.
Then

$$\|u\|_{L^{\frac{\alpha p}{n-p}}(\Omega,\mu)} \le C\big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^{\frac{p(n-1)}{n-p}}(\partial\Omega)}\big) \tag{2}$$

for every u with bounded support, where $C = C(n)$.

Example 2. In the borderline case when $p = n$, we obtain that

$$\|u\|_{\exp L^{\frac{n}{n-1}}(\Omega,\mu)} \le C\Big(\|\nabla u\|_{L^n(\Omega)} + \|u\|_{\exp L^{\frac{n}{n-1}}(\partial\Omega)}\Big), \qquad (3)$$

for some constant C and every function u with bounded support, provided that $\mathcal{L}^n(\Omega) < \infty$, $\mu(\Omega) < \infty$ and $\mathcal{H}^{n-1}(\partial\Omega) < \infty$.

Here, $\|\cdot\|_{\exp L^{\frac{n}{n-1}}(\Omega,\mu)}$ and $\|\cdot\|_{\exp L^{\frac{n}{n-1}}(\partial\Omega)}$ denote norms in Orlicz spaces of exponential type on Ω and $\partial\Omega$, respectively.

Inequality (3) extends the Yudovich-Pohozaev-Trudinger inequality to possibly irregular domains.

It also improves a result of [Maggi-Villani, 2008], where estimates for the weaker norm in $\exp L(\Omega)$ are established, and just for the Lebesgue measure.

Our main focus is on higher-order inequalities.

New seminorms on the boundary are introduced.

Prototypal case: second-order Sobolev inequalities ($m = 2$).

A notion of "upper gradient " for functions defined on $\partial\Omega$, regarded as a measure spaces, comes into play.

An upper gradient for the trace of u on $\partial\Omega$ is any Borel function $g : \partial\Omega \to [0, \infty]$ s.t.

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x, y \in \partial\Omega.$$

This notion, with $\partial\Omega$ replaced with any metric measure space, was introduced in [Hajlasz, 1996] to define Sobolev spaces on arbitrary metric measure spaces.

It extends the standard notion of weak gradient for functions u defined in $\mathbb{R}^n$, since, in this case,

$$|u(x) - u(y)| \leq C|x - y|(M(|\nabla u|)(x) + M(|\nabla u|)(y)) \quad \text{for a.e. } x, y \in \mathbb{R}^n,$$

where M denotes the maximal function operator.

This, and alternative definitions of upper gradients for functions defined on metric measure spaces, have been investigated and applied in a rich literature in the last two decades.

Given $r \in [1, \infty]$, we define the seminorm

$$\|u\|_{\mathcal{V}^{1,0}L^r(\partial\Omega)} = \inf_g \|g\|_{L^r(\partial\Omega)},$$

where the infimum is taken among all upper gradients of u over $\partial\Omega$.

Here, as above, $\|\cdot\|_{L^r(\partial\Omega)}$ denotes a Lebesgue norm on $\partial\Omega$ with respect to the $(n-1)$-dimensional Hausdorff measure.

Analogous seminorms, with $L^r(\partial\Omega)$ replaced with other function norms on $\partial\Omega$, are defined accordingly.

Let Ω be any open set in $\mathbb{R}^n$.

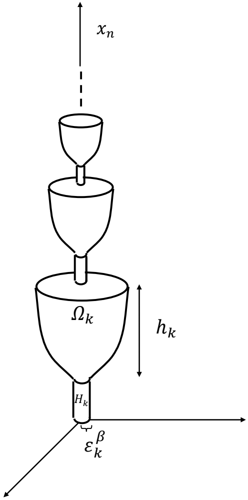
Example 3. Estimate for u by $\nabla^2 u$.
If $1 < p < \frac{n}{2}$, then

$$\|u\|_{L^{\frac{pn}{n-2p}}(\Omega)} \leq C\big(\|\nabla^2 u\|_{L^p(\Omega)} + \|u\|_{\mathcal{V}^{1,0}L^{\frac{p(n-1)}{n-p}}(\partial\Omega)} + \|u\|_{L^{\frac{p(n-1)}{n-2p}}(\partial\Omega)}\big). \tag{4}$$

- $\frac{pn}{n-2p}$ is the critical Sobolev exponent for $m = 2$, same as for regular Ω.
- $C = C(p, n)$, independent of Ω.

If Ω is regular, then the term $\|u\|_{\mathcal{V}^{1,0}L^{\frac{p(n-1)}{n-p}}(\partial\Omega)}$ can be dropped in (4), but then C depends also on Ω.

Counterexamples show that this is not possible in an arbitrary domain.



Example 4. Estimate for ∇u by $\nabla^2 u$

If $1 < p < n$ and $r \geq 1$, then

$$\|\nabla u\|_{L^q(\Omega)} \leq C\big(\|\nabla^2 u\|_{L^p(\Omega)} + \|u\|_{\mathcal{V}^{1,0}L^r(\partial\Omega)}\big). \tag{5}$$

Here,

$$q = \min\{\tfrac{rn}{n-1}, \tfrac{np}{n-p}\}.$$

The constant C in (5) depends only on n and p if $r = \frac{p(n-1)}{n-p}$ and $q = \frac{np}{n-p}$. Otherwise, it also depends on the Lebesgue measure $\mathcal{L}^n(\Omega)$.

Examples show that the exponent q is optimal, if no regularity on Ω is imposed.

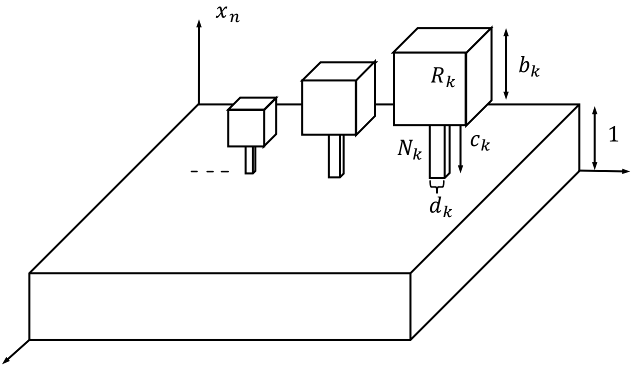


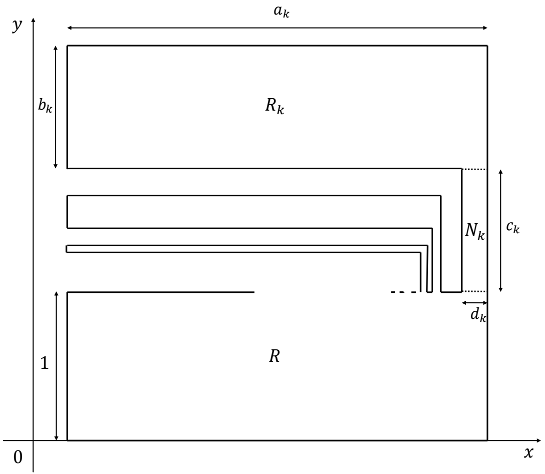
Figure:

If Ω is regular, the seminorm $\|u\|_{\mathcal{V}^{1,0}L^r(\partial\Omega)}$ can be replaced with $\|u\|_{L^r(\partial\Omega)}$ in the previous inequality.
Namely, $\forall \;\; r \geq 1$,

$$\|\nabla u\|_{L^q(\Omega)} \leq C\big(\|\nabla^2 u\|_{L^p(\Omega)} + \|u\|_{L^r(\partial\Omega)}\big) \tag{6}$$

with $q = \frac{np}{n-p}$, for some constant $C = C(n, p, r, \Omega)$.

Counterexamples show that inequality (6) may fail on an arbitrary domain.



Major motivation for our research: provide a functional framework for boundary value problems for PDE's, and variational problems, in domains lacking any regularity.
Consider, as a simple instance, the problem

$$\begin{cases} \Delta^2 u = \operatorname{div} F & \text{in } \Omega \\ u = 0, \ \mathcal{B}\, u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where:

- Ω is any open set in $\mathbb{R}^n$, $n \geq 3$,
- Δ^2 is the bi-Laplace operator,
- $F : \Omega \to \mathbb{R}^n$ is a given function,
- $\mathcal{B}$ is the (second-order) boundary operator generated by the minimization of the quadratic functional

$$\int_\Omega \left(|\nabla^2 u|^2 + 2F \cdot \nabla u \right) dx$$

among functions u vanishing on $\partial\Omega$.

Assume that $F \in L^{\frac{2n}{n+2}}(\Omega)$, where $\frac{2n}{n+2}$ is the Hölder conjugate of the critical Sobolev exponent $\frac{2n}{n-2}$.

Inequality (5) discussed above tells us that $\exists\ C = C(n)$ such that

$$\|\nabla u\|_{L^{\frac{2n}{n-2}}(\Omega)} \le C\|\nabla^2 u\|_{L^2(\Omega)} \tag{8}$$

$\forall$ function u vanishing on $\partial\Omega$.

By Riesz' representation theorem in Hilbert spaces, $\exists!$ solution u to the boundary value problem (7) for the bi-Laplace operator, whatever Ω is.

Incidentally, notice that, instead, inequality (8) fails if $\nabla^2 u$ is replaced just with Δu, unless Ω is sufficiently regular.

Main results for arbitrary $m \in \mathbb{N}$.

Given $m \in \mathbb{N}$ and $p \in [1,\infty]$, denote by $V^{m,p}(\Omega)$ the Sobolev type space defined as

$$V^{m,p}(\Omega) = \big\{u : u \text{ is } m\text{-times weakly diff. in } \Omega, \text{ and } |\nabla^m u| \in L^p(\Omega)\big\}.$$

Notice that, in the definition of $V^{m,p}(\Omega)$, it is only required that the derivatives of the highest order m of u belong to $L^p(\Omega)$.

For $k \in \mathbb{N}_0$, denote as usual by $C^k(\overline{\Omega})$ the space of functions whose k-th order derivatives in Ω are continuous up to the boundary.

Also set

$$C^k_{\mathrm{b}}(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}) : u \text{ has bounded support}\}. \tag{9}$$

Clearly,

$$C^k_{\mathrm{b}}(\overline{\Omega}) = C^k(\overline{\Omega}) \quad \text{if } \Omega \text{ is bounded.}$$

- Pointwise estimates.

A higher-order notion of the upper gradient involves a kind of higher-order difference quotients.

If $k \in \mathbb{N}$, denote by $g^{k,0}$ any Borel function on $\partial\Omega$ s.t.

$$\Big| \sum_{|\alpha| \leq k-1} \frac{(2k-2-|\alpha|)!}{(k-1-|\alpha|)!\alpha!} \frac{(y-x)^\alpha}{|y-x|^{2k-1}} \Big[(-1)^{|\alpha|} D^\alpha u(y) - D^\alpha u(x)\Big] \Big| \leq g^{k,0}(x) + g^{k,0}(y)$$

for $\mathcal{H}^{n-1}$-a.e. $x, y \in \partial\Omega$.

If $k \in \mathbb{N}$, denote by $g^{k,1}$ any Borel function on $\partial\Omega$ s.t.

$$\sum_{i=1}^{n} \Big| \sum_{|\alpha| \le k-1} \frac{(2k-2-|\alpha|)!}{(k-1-|\alpha|)!\alpha!} \frac{(y-x)^\alpha}{|y-x|^{2k-1}} \Big[(-1)^{|\alpha|} D^\alpha u_{x_i}(y) - D^\alpha u_{x_i}(x) \Big] \Big| \le g^{k,1}(x) + g^{k,1}(y)$$

for $\mathcal{H}^{n-1}$-a.e. $x, y \in \partial\Omega$.

Also, $g^{0,1}$ denotes any Borel function on $\partial\Omega$ s.t.

$$|u(x)| \le g^{0,1}(x) \quad \text{for } x \in \partial\Omega.$$

As in the basic case of the (first-order) upper gradient, these definitions extend standard notions for functions on $\mathbb{R}^n$.

Indeed, if $n, k \in \mathbb{N}$ and $u \in W^{2k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$\left| \sum_{|\alpha| \le k-1} \frac{(2k-2-|\alpha|)!}{(k-1-|\alpha|)!\alpha!} \frac{(y-x)^\alpha}{|y-x|^{2k-1}} \left[(-1)^{|\alpha|} D^\alpha u(y) - D^\alpha u(x)\right] \right| \le C\big(M(|\nabla^{2k-1} u|)(x) + M(|\nabla^{2k-1} u|)(y)\big)$$

for a.e. $x, y \in \mathbb{R}^n$.

Further notation.

Let

$$\zeta : \Omega \times \mathbb{S}^{n-1} \to \mathbb{R}^n$$

be the function which associates with each $(x, \vartheta) \in \Omega \times \mathbb{S}^{n-1}$ the first point $\zeta(x, \vartheta) \in \partial\Omega$ of intersection of the half-line $x + t\vartheta$, $t \geq 0$, with $\partial\Omega$ (if the intersection is not empty).

For $k \in \mathbb{N}$, we set

$$\natural(k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

Theorem 1: Pointwise estimates for $|\nabla^h u|$

Let Ω be any open set in $\mathbb{R}^n$, $n \geq 2$, and let $m \in \mathbb{N}$, and $h \in \mathbb{N}_0$ be s.t. $0 < m-h < n$. Then $\exists\ C = C(n,m)$ s.t.

$$|\nabla^h u(x)| \leq C\Big(\int_\Omega \frac{|\nabla^m u(y)|}{|x-y|^{n-m+h}}\, dy + \sum_{k=1}^{m-h-1} \int_\Omega \int_{\mathbb{S}^{n-1}} \frac{g^{[\frac{k+h+1}{2}],\natural(k+h)}(\zeta(y,\vartheta))}{|x-y|^{n-k}}\, d\mathcal{H}^{n-1}(\vartheta)\, dy + \int_{\mathbb{S}^{n-1}} g^{[\frac{h+1}{2}],\natural(h)}(\zeta(x,\vartheta))\, d\mathcal{H}^{n-1}(\vartheta) \Big) \qquad \text{for a.e. } x \in \Omega,$$

$\forall\ \ u \in V^{m,1}(\Omega) \cap C_{\mathrm{b}}^{[\frac{m-1}{2}]}(\overline{\Omega})$.

Remark 1. In the case when $m-h=n$, an analogous estimate holds, with the kernel $\dfrac{1}{|x-y|^{n-m+h}}$ in the first integral on the right-hand side replaced with $\log\dfrac{C}{|x-y|}$.

Remark 2. If

$$u=\nabla u=\ldots\nabla^{[\frac{m-1}{2}]}u=0 \quad \text{on} \quad \partial\Omega,$$

the estimate reduces to

$$|\nabla^h u(x)|\le C\int_\Omega \frac{|\nabla^m u(y)|}{|x-y|^{n-m+h}}\,dy \qquad \text{for a.e. } x\in\Omega.$$

- Rearrangement estimates.

A new kind of double-integral operators appear in our estimates.
In order to derive norm inequalities, their boundedness properties have to be investigated.

Flexible approach: obtain rearrangement estimates for u or $|\nabla^h u|$ in terms of $|\nabla^m u|$ and $g^{k,j}$.

We can deal with norms of u or $|\nabla^h u|$ with respect to measures μ s.t.

$$\mu(B_r(x) \cap \Omega) \leq C_\mu r^\alpha \quad \text{for } x \in \Omega,$$

for some $\alpha \in (n-1, n]$ and $C_\mu > 0$.

In particular, if $\mu = \mathcal{L}^n$, then $\alpha = n$.

Recall that given a positive measure space $\mathcal{R}$, endowed with a measure ν, and a ν-measurable function $\phi : \mathcal{R} \to \mathbb{R}$, the decreasing rearrangement $\phi^*_\nu : [0, \infty) \to [0, \infty]$ is defined as

$$\phi^*_\nu(s) = \inf\{t \in \mathbb{R} : \nu(\{|u| > t\}) \le s\} \quad \text{for } s \in [0, \infty).$$

Thus,

$$\nu(\{\phi > t\}) = \mathcal{L}^1(\{\phi^*_\nu > t\}) \quad \forall t > 0.$$

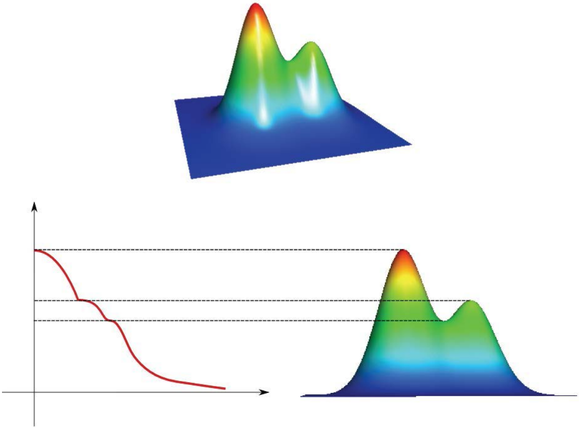


FIGURE:

Given any rearrangement-invariant norm $\| \cdot \|_{X(\mathcal{R},\nu)}$,

$$\|\phi\|_{X(\mathcal{R},\nu)} = \|\phi_\nu^*\|_{\overline{X}(0,\infty)} \qquad \forall\ \phi,$$

where $\overline{X}(0,\infty)$ denotes the representation space of $X(\mathcal{R},\nu)$.

As a consequence, rearrangement inequalities reduce the problem of n-dimensional Sobolev type inequalities in arbitrary open sets to considerably simpler one-dimensional Hardy type inequalities.

Theorem 2: Rearrangement estimate for $|\nabla^h u|$

Let Ω be any open set in $\mathbb{R}^n$, $n \geq 2$. Assume that $\mu(B_r(x) \cap \Omega) \leq C_\mu r^\alpha$ with $\alpha \in (n-1, n]$. Let $m \in \mathbb{N}$, and $h \in \mathbb{N}_0$ be s.t. $0 < m - h < n$. Then $\exists \quad C = C(n, m, \alpha, C_\mu)$ s.t., $\forall\ u \in V^{m,1}(\Omega) \cap C_{\mathrm{b}}^{[\frac{m-1}{2}]}(\overline{\Omega})$,

$$
\begin{aligned}
|\nabla^h u|^*_\mu(cs) \leq C \Bigg[& s^{-\frac{n-m+h}{\alpha}} \int_0^{s^{\frac{n}{\alpha}}} |\nabla^m u|^*_{\mathcal{L}^n}(r)\,dr \\
& + \int_{s^{\frac{n}{\alpha}}}^\infty |\nabla^m u|^*_{\mathcal{L}^n}(r) r^{-\frac{n-m+h}{n}}\,dr \\
& + \sum_{k=1}^{m-h-1} \Bigg(s^{-\frac{n-1-k}{\alpha}} \int_0^{s^{\frac{n-1}{\alpha}}} \left[g^{[\frac{k+h+1}{2}],\natural(k+h)}\right]^*_{\mathcal{H}^{n-1}}(r)\,dr \\
& + \int_{s^{\frac{n-1}{\alpha}}}^\infty \left[g^{[\frac{k+h+1}{2}],\natural(k+h)}\right]^*_{\mathcal{H}^{n-1}}(r) r^{-\frac{n-1-k}{n-1}}\,dr \Bigg) \\
& + s^{-\frac{n-1}{\alpha}} \int_0^{s^{\frac{n-1}{\alpha}}} \left[g^{[\frac{h+1}{2}],\natural(h)}\right]^*_{\mathcal{H}^{n-1}}(r)\,dr \Bigg] \quad \text{for } s > 0.
\end{aligned}
$$

- Sobolev inequalities.

Given a rearrangement-invariant norm $\| \cdot \|_{X(\partial\Omega)}$, $k \in \mathbb{N}$ and $j = 0, 1$, define

$$\|u\|_{\mathcal{V}^{k,j}X(\partial\Omega)} = \inf_{g^{k,j}} \|g^{k,j}\|_{X(\partial\Omega)},$$

where $g^{k,j}$ ranges among all higher-order upper gradients.

Theorem 3: Sobolev inequality with measure

Let Ω be any open bounded open set in $\mathbb{R}^n$, $n \geq 2$. Assume that μ is a measure in Ω s.t.

$$\mu(B_r(x) \cap \Omega) \leq C_\mu r^\alpha$$

for some $\alpha \in (n-1, n]$. Let $m \in \mathbb{N}$, and $h \in \mathbb{N}_0$ be such that $0 < m-h < n$. If $1 < p < \frac{n}{m-h}$, then $\exists \;\; C = C(n, m, p, \alpha, C_\mu)$ s.t.

$$\|\nabla^h u\|_{L^{\frac{\alpha p}{n-(m-h)p}}(\Omega,\mu)} \leq C\|\nabla^m u\|_{L^p(\Omega)} + C \sum_{k=0}^{m-h-1} \|u\|_{\mathcal{V}^{[\frac{k+h+1}{2}],\natural(k+h)} L^{\frac{p(n-1)}{n-(m-h-k)p}}(\partial\Omega)}$$

$\forall \;\; u \in V^{m,p}(\Omega) \cap C_{\mathrm{b}}^{[\frac{m-1}{2}]}(\overline{\Omega})$.

As in the classical Rellich theorem, the Sobolev embedding corresponding to the inequality of the last theorem is pre-compact if the exponent $\frac{\alpha p}{n-(m-h)p}$ is replaced with a smaller one, and the measure of the domain is finite.

Theorem 4: Compact Sobolev embedding with measure

Let Ω, μ, n, m and h be as in the preceding Theorem. Assume, in addition, that $\mu(\Omega) < \infty$. If $1 \leq q < \frac{\alpha p}{n-(m-h)p}$, then the embedding

$$V^{m,p}(\Omega) \cap C_{\mathrm{b}}^{[\frac{m-1}{2}]}(\overline{\Omega}) \to V^{h,q}(\Omega, \mu),$$

is pre-compact.

The limiting case when $p = \frac{n}{m-h}$ is the object of the next result, which provides us with a Pohozaev-Trudinger-Yudovich type inequality in arbitrary domains.

In the statement,

$$\|\phi\|_{\exp L^\alpha(\mathcal{R},\nu)} = \inf\left\{\lambda \geq 0 : \int_{\mathcal{R}} e^{(|\phi|/\lambda)^\alpha} - 1 \; d\nu \leq 1\right\},$$

and

$$\|\phi\|_{L^p(\log L)^\beta(\mathcal{R},\nu)} = \inf\left\{\lambda \geq 0 : \int_{\mathcal{R}} (|\phi|/\lambda)^p (1 + \log(|\phi|/\lambda))^\beta \; d\nu \leq 1\right\}.$$

Theorem 5: Limiting Sobolev inequality with measure

Let Ω be any open bounded open set in $\mathbb{R}^n$, $n \geq 2$. Assume that μ is a measure in Ω s.t.

$$\mu(B_r(x) \cap \Omega) \leq C_\mu r^\alpha$$

for some $\alpha \in (n-1, n]$. Assume, in addition, that $\mathcal{L}^n(\Omega), \mu(\Omega), \mathcal{H}^{n-1}(\partial\Omega) < \infty$. Let $m \in \mathbb{N}$ and $h \in \mathbb{N}_0$ be such that $0 < m-h < n$. Then $\exists$ C s.t.

$$\begin{aligned} \|\nabla^h u\|_{\exp L^{\frac{n}{n-(m-h)}}(\Omega,\mu)} &\leq C\|\nabla^m u\|_{L^{\frac{n}{m-h}}(\Omega)} \\ &+ C \sum_{k=1}^{m-h-1} \|u\|_{\mathcal{V}^{[\frac{k+h+1}{2}],\natural(k+h)} L^{\frac{n-1}{k}}(\log L)^{\frac{(m-h)(n-k-1)}{nk}}(\partial\Omega)} \\ &+ C\|u\|_{\mathcal{V}^{[\frac{h+1}{2}],\natural(h)} \exp L^{\frac{n}{n-(m-h)}}(\partial\Omega)} \end{aligned}$$

$\forall \;\; u \in V^{m,\frac{n}{m-h}}(\Omega) \cap C_{\mathrm{b}}^{[\frac{m-1}{2}]}(\overline{\Omega})$.

The super-limiting regime when $p > \frac{n}{m-h}$ is the object of the following result.

Theorem 6: Super-limiting Sobolev inequality

Let Ω be a open set in $\mathbb{R}^n$, $n \geq 2$, such that $\mathcal{L}^n(\Omega), \mathcal{H}^{n-1}(\partial\Omega) < \infty$. Assume that $m \in \mathbb{N}$, $h \in \mathbb{N}_0$, and $0 < m - h < n$. If $p > \frac{n}{m-h}$ and $p_k > \frac{n-1}{k}$ for $k = 1, \ldots, m-h-1$, then $\exists \quad C$ s.t.

$$\begin{aligned} \|\nabla^h u\|_{L^\infty(\Omega)} &\leq C\|\nabla^m u\|_{L^p(\Omega)} \\ &\quad + C \sum_{k=1}^{m-h-1} \|u\|_{\mathcal{V}^{[\frac{k+h+1}{2}],\natural(k+h)} L^{p_k}(\partial\Omega)} \\ &\quad + C\|u\|_{\mathcal{V}^{[\frac{h+1}{2}],\natural(h)} L^\infty(\partial\Omega)} \end{aligned}$$

$\forall \quad u \in V^{m,p}(\Omega) \cap C_b^{[\frac{m-1}{2}]}(\overline{\Omega})$.

More inequalities, involving other optimal target norms, can be obtained.

For instance, inequalities in $V^{m,p}(\Omega)$, $1 < p < \frac{n}{m}$, with the target norm $\|u\|_{L^{\frac{np}{n-mp}}(\Omega)}$ replaced with the stronger Lorentz norm $\|u\|_{L^{\frac{np}{n-mp},p}(\Omega)}$.

In the borderline case when $p = \frac{n}{m}$, Hansson-Brezis-Weinger type inequalities also follow, and improve the exponential target norm by a Lorentz-Zygmund norm.