Complex Jacobi matrices and uniqueness for discrete Schrödinger evolutions

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Tribute to Victor Havin (1933-2015)

Discrete Schrödinger evolutions

Main Equation

$$\partial_t u = i(\Delta_d u + Vu),$$

where $u : \mathbb{R}_+ \times \mathbb{Z}^d \to \mathbb{C}$ and Δ_d is the standard discrete Laplacian. For $d = 1$,

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n).$$

We assume that the potential $V = V(t, n)$ is a bounded complex-valued function.

Related Equation

$$\partial_t u = \alpha(\Delta_d u + Vu),$$

where $\alpha \in \mathbb{C}$ is a constant.

Uniqueness

$$|u(0,n)| + |u(1,n)| \leq Cm(n) \quad \Rightarrow \quad u \equiv 0.$$

Continuous case

L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega (and M. Cowling) (2006-12)

$$\partial_t u = i\Delta u, \quad |u(0,x)| + |u(1,x)| \le C\exp(-x^2/4),$$
$$(*) \qquad \Rightarrow \quad u(0,x) = A\exp(-(1+i)x^2/4)$$

For any bounded $V(x,t)$ and any $a > 1/4$

$$\partial_t u = i\Delta u + Vu, \quad |u(0,x)| + |u(1,x)| \le C\exp(-ax^2),$$

$$(**) \qquad \Rightarrow \quad u(t,x) \equiv 0$$

Free evolution and uncertainty principles

Hardy's uncertainty principle:

$$|f(x)| \leq C \exp(-\pi|x|^2), \quad |\hat{f}(\xi)| \leq C \exp(-\pi|\xi|^2),$$

$$\Rightarrow \quad f(x) = A \exp(-\pi|x|^2)$$

is equivalent to (*)

Heisenberg's uncertainty principle can be reformulated in terms of

$$h(t) = \|xu(t,x)\|_2$$

Logarithmic convexity for weighted norms of solutions to PDE

Elliptic PDE: S. Agmon (1966); Landis and others (1980s), Garofalo and Lin (1987), Brummelhuis (1995)

Schrödinger equation: Escauriaza, Kenig, Ponce, Vega

$$H_R(t) = \|\phi_R(x)u(t,x)\|_2^2, \quad \phi_R(x) = \exp(\gamma|x + Rt(1-t)|^2)$$

$$\partial_t^2 \log H_R(t) \geq -R^2(4\gamma)^{-1}$$

$$\exp(-R^2(16\gamma)^{-1})H_R(1/2) \leq H_R(0)^{1/2}H_R(1)^{1/2} = H(0)^{1/2}H(1)^{1/2}$$

Let $R \to \infty$ and get a contradiction when $\gamma > \gamma_0$.

Discrete case: heat kernel, log-convexity and uncertainty

Chang and Yau, 1997, 2000

Three spheres theorem and logarithmic convexity for weighted norms of discrete harmonic functions:
Gaudi and Malinnikova (Compt. Methods and Function Th, 2014)
Lippner and Mangoubi (Duke Math. J., 2015)

Heisenberg's uncertainty, interpretation for discrete Schrödinger evolution: Fernández-Bertolin (ACHA, 2016)

Previous result: uniqueness

Theorem (Jaming, Lyubarskii, M, Perfekt arXiv2015, to appear in RMI; Fernandez-Bertolin, Vega, arXiv2015)

If u is a solution of

$$\partial_t u = i(\Delta_d u + Vu)$$

where $V(t,n)$ is a bounded function, and

$$\|(1+|n|)^{\gamma(1+|n|)}u(0,n)\|_2, \|(1+|n|)^{\gamma(1+|n|)}u(1,n)\|_2 < +\infty,$$

for $\gamma > \gamma_0$, then $u \equiv 0$.

Our approach: logarithmic convexity of the norms

$$H(t) = \|\psi(n)u(t,n)\|_2$$

for some appropriate $\psi \Rightarrow$ uniqueness.

Toy example: Free discrete Schrödinger

Proposition

Let $\partial_t u = i\Delta_d u$, *and*

$$|u(0,n)|,\ |u(1,n)| \leq C\frac{1}{\sqrt{|n|}}\left(\frac{e}{2|n|}\right)^{|n|} \sim J_n(1).$$

Then $u(t,n) = Ai^{-n}e^{-2it}J_n(1-2t)$ *for all* $n \in \mathbb{Z}$ *and* $0 \leq t \leq 1$, *for some constant* A.

The Bessel functions are defined by

$$\exp(x(z - z^{-1})/2) = \sum_{n=-\infty}^{\infty} J_n(x)z^n$$

and satisfy

$$|J_n(x)| \sim \frac{1}{\sqrt{|n|}}\left(\frac{ex}{2|n|}\right)^{|n|}$$

Entire functions of exponential type

Let f be an entire function such that

$$\sigma = \limsup_{r\to\infty} \frac{\log \max\{|f(z)|, |z| = r\}}{r} < \infty$$

We consider the indicator function of f

$$h_f(\phi) = \limsup_{r\to\infty} \frac{\log|f(re^{i\phi})|}{r}$$

Then h_f is the support function of a convex set I_f and, in particular,

$$h_f(\phi) + h_f(\phi + \pi) \geq 0$$

If an entire function of exponential type does not grow fast the it does not decay fast!

Toy example: Proof

Consider

$$\Phi(t,\theta) = \sum_{k=-\infty}^{\infty} u(t,k)\theta^k \ \in L^2(\mathbb{T})$$

Then

$$\partial_t \Phi(t,\theta) = i(\theta + \theta^{-1} - 2)\Phi(t,\theta),$$

Integrated,

$$\Phi(1,\theta) = e^{i(\theta+\theta^{-1}-2)}\Phi(0,\theta)$$

Thus $I_0 = I_{\Phi(0)}$ and $I_1 = I_{\Phi(1)}$ satisfy $I_1 = I_0 + i$.
On the other hand, the estimates of the coefficients give $I_1, I_0 \subset \frac{1}{2}\mathbb{D}$. Hence $I_1 = i/2$ and $I_0 = -i/2$.

Finally, $g(z) = \exp(i(z + z^{-1})/2)\Phi(0,z)$ is of zero type and bounded on the imaginary axis. By the Phragmen-Lindelöf theorem, $g = \mathit{constant}$.

Remarks

- Schrödinger and heat evolutions can be done by the same approach.
- One-sided estimate is enough: if $\partial_t u = i\Delta_d u$ and $|u(t,n)| \le Ce^n(2+\varepsilon)^{-n}n^{-n}$ for $n > 0$ and $t = 0, 1$ then $u = 0$.
- It generalizes to higher dimensions with the condition $\sum_{n\in\mathbb{Z}^d : n_d = m} |u(t,n)|^2 \le e^m(2+\varepsilon)^{-m}m^{-m}$.
- The same proof works for perturbed equation with time-independent small potential $V = V_n$, Jost solutions. We consider similar functions $\Phi(t,\theta) = \sum_n u(t,n)e_n(\theta)$, show that $\Phi = 0$ and conclude that $u = 0$.

Time-independent real potentials

Proposition (Lyubarskii, M, 2016)

Let $u(t,n)$, $t>0$, $n\in\mathbb{Z}$ be a solution of $\partial_t u = i(\Delta u + Vu)$, where $V=V_n$ is bounded and real-valued. If, for some $\varepsilon>0$,

$$|u(t,n)| \le C\left(\frac{e}{(2+\varepsilon)|n|}\right)^{|n|}, \quad n\in\mathbb{Z},\ t\in\{0,1\},$$

then $u=0$.

Outline of proof

Generalized eigenvectors $e_n(\lambda)$ of $\Delta_d + V$: components grow not faster than C^n.
Consider $\Psi(\lambda,t)=\sum_n u(t,n)e_n(\lambda)$ as before, by the Phragmén-Lindelöf theorem $\Psi(\lambda,0)=0$.
Spectral theorem for self-adjoint operators implies that if $\{u(0,n)\}$ is orthogonal to all generalized eigenvectors then it is zero. (Gelfand, Shilov and Gelfand, Vilenkin).

Complex Jacobi matrices and orthogonal polynomials

Let $P_n(z)$ be polynomials such that $P_0 = 1$, $P_1(z) = (z - a_1)/b_1$ and

$$zP_n(z) = b_{n+1}P_{n+1}(z) + a_{n+1}P_n(z) + b_nP_{n-1}(z), \;\; n \geq 1,$$

where $a_n, b_n \in \mathbb{C}$, $b_n \neq 0$. Then there exists a linear functional L on the space of all polinomials such that $L(P_n\overline{P_m}) = 0$ when $n \neq m$ and $L(P_n^2) = 1$ (L is defined by $L(1) = 1$ and $L(P_n) = 0$). Moreover, when a_n and b_n are bounded we get

$$|L(z^n)| \leq C^n.$$

Suppose that $u(n, 0)$ is such that $u(n, 0)$ tends to zero fast and $\sum_n u(n, 0)P_n(z) = 0$ for any z. Then

$$0 = \lim_{N\to\infty} L\left(\sum_{n\leq N} z^k u(n, 0)P_n(z)\right) = b_1...b_k u(k, 0)$$

Time-independent bounded complex-valued potentials

Theorem (Lyubarskii, M, 2016)

Let $u(t,n)$, $t>0$, $n\in\mathbb{Z}$ be a solution of $\partial_t u = i(\Delta u + Vu)$, where $V=V_n$ is bounded. If, for some $\varepsilon>0$,

$$|u(t,n)| \le C\left(\frac{e}{(2+\varepsilon)|n|}\right)^{|n|}, \quad n\in\mathbb{Z},\ t\in\{0,1\},$$

then $u=0$.

As before $\sum_n u(n,0)P_n(z)=0$ for any sequence of polynomials satisfying the recurrent relation (and for any z). We take two sequences of polynomials $\{P_n\}_{n\in\mathbb{Z}}$ and $\{Q_n\}_n \in \mathbb{Z}$ and obtain $u(n,0)\equiv 0$.

Remarks and open problems

- There is a multidimensional version of the result for the case of real valued potential, it is not clear if the corresponding result holds for complex potentials.
- Sharp uniqueness results for time-dependent real-valued potentials holds for heat equation.
- Our computation suggest that for the case of the Schrödinger operator and a complex time-dependent potential, one should look for a counterexample.
- It would be interesting to consider the discrete and continuous cases simultaneously.