The norm-preserving extension property in the symmetrized bidisc Γ and von Neumann-type inequalities for Γ-contractions

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The norm-preserving extension property

A celebrated theorem of H. Cartan states.

Theorem 1. *If V is an analytic variety in a domain of holomorphy U and if f is an holomorphic function on V then there exists an holomorphic function on U whose restriction to V is f.*

In the case that the function f is bounded, Cartan's theorem gives no information on the supremum norm of any holomorphic extension of f. We are interested in the case that an extension exists with the *same* supremum norm as f.

Definition 1. *A subset V of a domain $U \subset \mathbb{C}^n$ has the* norm-preserving extension property *if, for every bounded holomorphic function f on V, there exists an holomorphic function g on U such that*

$$g|V = f \quad \textit{and} \quad \sup_U |g| = \sup_V |f|.$$

Here V is not assumed to be a variety; to say that f is holomorphic on V simply means that, for every $z \in V$ there exists a neighborhood W of z in U and an holomorphic function h on W such that h agrees with f on $W \cap V$.

Retracts and geodesics

Let U be a domain. We say that ρ is a *retraction of* U if $\rho \in \mathrm{Hol}(U,U)$ and $\rho \circ \rho = \rho$. A set $R \subseteq U$ is a *retract in* U if there exists a retraction ρ of U such that $R = \operatorname{ran} \rho$.

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We say that a retraction ρ and the corresponding retract R are *trivial* if either ρ is constant (and R is a singleton set) or $\rho = \mathrm{id}_U$ (and $R = U$).

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Definition 2. *Let $\mathcal{D} \subset U$. We say that $\mathcal{D}$ is a* complex geodesic *in U if there exists a function $k \in \mathrm{Hol}(\mathbb{D}, U)$ and a function $C \in \mathrm{Hol}(U, \mathbb{D})$ such that $C \circ k = \mathrm{id}_{\mathbb{D}}$ and $\mathcal{D} = k(\mathbb{D})$.*

Here and below $\mathbb{D}$ is open unit disc in the complex plane $\mathbb{C}$, $\overline{\mathbb{D}}$ is closed unit disc in $\mathbb{C}$, and $\mathrm{Hol}(U, \Omega)$ is the set of holomorphic mappings from U to Ω.

General connections

For a subset V of a domain U, the statements

i. V is a singleton, a complex geodesic or all of U,

ii. V is a retract in U,

iii. V has the norm-preserving extension property in U

satisfy (i) implies (ii) implies (iii).

V in $\mathbb{D}^2$ with the norm-preserving extension property

It was shown by Jim Agler and John McCarthy in [1] that, in the case that U is the bidisc $\mathbb{D}^2$, the converse implications also hold.

Theorem 2. [Agler and McCarthy, 2003] *An algebraic set V in $\mathbb{D}^2$ has the norm-preserving extension property if and only if V has one of the following forms.*

i. $V = \{\lambda\}$ *for some* $\lambda \in \mathbb{D}^2$*;*

ii. $V = \mathbb{D}^2$*;*

iii. $V = \{(z, f(z)) : z \in \mathbb{D}\}$ *for some* $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$*;*

iv. $V = \{(f(z), z) : z \in \mathbb{D}\}$ *for some* $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$*.*

We say that a set V in $\mathbb{C}^2$ is an *algebraic set* if there exists a set S of polynomials in two variables such that

$$V = \{\lambda \in \mathbb{C}^2 : p(\lambda) = 0 \text{ for all } p \in S\}.$$

Retracts and geodesics of the symmetrised bidisc

Jim Agler and Nicholas Young began the study of the *open symmetrised bidisc*

$$\mathbb{G} \stackrel{\text{def}}{=} \{(z+w, zw) : z, w \in \mathbb{D}\} \subset \mathbb{C}^2,$$

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One might conjecture that $(\mathrm{i}) \iff (\mathrm{ii}) \iff (\mathrm{iii})$ for a general domain in $\mathbb{D}^2$; however, for the symmetrized bidisc,

(ii) $\iff$ (i), but (iii) does not imply (ii).

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(ii) $\iff$ (i), but (iii) does not imply (ii).

Theorem 3. *A map $\rho \in \mathrm{Hol}(G, G)$ is a nontrivial retraction of G if and only if $\rho = k \circ C$ for some $C \in \mathrm{Hol}(G, \mathbb{D})$ and $k \in \mathrm{Hol}(\mathbb{D}, G)$ such that $C \circ k = \mathrm{id}_{\,\mathbb{D}}$. Thus R is a nontrivial retract in G if and only if R is a complex geodesic in G.*

Theorem 4. *For every complex geodesic $\mathcal{D}$ in G there exists a polynomial P of total degree at most 2 such that $\mathcal{D} = \{s \in G : P(s) = 0\}$.*

A classification of geodesics in G

We say that two geodesics $\mathcal{D}_1, \mathcal{D}_2$ are equivalent (written $\mathcal{D}_1 \sim \mathcal{D}_2$) if there exists an automorphism $\tilde{m}$ of G such that $\tilde{m}(\mathcal{D}_1) = \mathcal{D}_2$.

Theorem 5. *Let $\mathcal{D}$ be a complex geodesic of G.*

i. $\mathcal{D}$ is purely unbalanced if and only if $\mathcal{D} \sim k_r(\mathbb{D})$ for some $r \in (0,1)$ where

$$k_r(z) = \frac{1}{1-rz}\left(2(1-r)z, z(z-r)\right) \qquad \text{for all } z \in \mathbb{D};$$

ii. $\mathcal{D}$ is exceptional if and only if $\mathcal{D} \sim h_r(\mathbb{D})$ for some real number $r > 0$, where

$$h_r(z) = (z + m_r(z), z m_r(z)) \tag{1}$$

and

$$m_r(z) = \frac{(r-i)z + i}{r + i - iz} \qquad \text{for } z \in \mathbb{D}; \tag{2}$$

iii. $\mathcal{D}$ is purely balanced if $\mathcal{D} \sim g_r(\mathbb{D})$ for some $r \in (0,1)$, where

$$g_r(z) = (z + B_r(z), zB_r(z)) \qquad \text{for all } z \in \mathbb{D};$$

iv. $\mathcal{D}$ is flat if and only if $\mathcal{D} \sim k(\mathbb{D})$ where $k(z) = (0, z)$;

v. $\mathcal{D}$ is royal if and only if $\mathcal{D} \sim k(\mathbb{D})$ where $k(z) = (2z, z^2)$.

Moreover, in statements (1), (2) and (3), the corresponding geodesics are pairwise inequivalent for distinct values of r in the given range.

V in G with the norm-preserving extension property

Theorem 6. *V is an algebraic subset of G having the norm-preserving extension property if and only if*
either V is a retract in G or
$V = \mathcal{R} \cup \mathcal{D}$, where $\mathcal{R} = \{(2z, z^2) : z \in \mathbb{D}\}$ and $\mathcal{D}$ is a flat geodesic in G.

A *flat geodesic* of G is a complex geodesic of G which is the intersection of G with a complex line. It is the set

$$\mathcal{F}_\beta = f_\beta(\mathbb{D}) = \{(\beta + \bar{\beta}z, z) : z \in \mathbb{D}\}, \text{ for some } \beta \in \mathbb{D}. \tag{3}$$

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Sets of the form $\mathcal{R} \cup \mathcal{D}$ are not retracts of G, but nevertheless have the norm-preserving extension property.

Subsets V with the A-extension property

Agler and McCarthy also generalized the norm-preserving extension property as follows.

Definition 3. *Let Ω be a domain of holomorphy,*
V be a subset of Ω and
A be a collection of bounded holomorphic functions on V.
*Then V is said to have the A-*extension property *(relative to Ω) if,*
for every $f \in A$,
there is a bounded holomorphic function g on Ω such that

$$g|V = f \quad \textit{and} \ \sup_{\Omega} |g| = \sup_{V} |f|.$$

The sets V in $\mathbb{D}^2$ with the symmetric extension property

Let V be a symmetric algebraic set in $\mathbb{D}^2$
('symmetric' meaning that $(\lambda^1, \lambda^2) \in V$ implies that $(\lambda^2, \lambda^1) \in V$).

Let $H^\infty_{\mathrm{sym}}(V)$ denote the algebra of bounded holomorphic functions g on V which are symmetric, in the sense that $g(\lambda^1, \lambda^2) = g(\lambda^2, \lambda^1)$ for all $(\lambda^1, \lambda^2) \in V$.

We say that V has the *symmetric extension property* if V has the $H^\infty_{\mathrm{sym}}(V)$-extension property.

The symmetric extension property in $\mathbb{D}^2$ is closely related to the norm-preserving extension property in G. We shall denote by t the transposition map $t(\lambda^1, \lambda^2) = (\lambda^2, \lambda^1)$.

Lemma 1. *A symmetric subset V of $\mathbb{D}^2$ has the symmetric extension property if and only if $\pi(V)$ has the norm-preserving extension property in G.*

Here the symmetrization map $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$\pi(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{C}.$$

The sets V in $\mathbb{D}^2$ with the symmetric extension property

Recall that a balanced disc in $\mathbb{D}^2$ is a subset D of $\mathbb{D}^2$ having the form $D = \{(z, m(z)) : z \in \mathbb{D}\}$ for some $m \in \mathrm{Aut}\ \mathbb{D}$. Here $\mathrm{Aut}\ \mathbb{D}$ is the automorphism group of $\mathbb{D}$.

Theorem 7. *A symmetric algebraic set V in $\mathbb{D}^2$ has the symmetric extension property if and only if one of the following six alternatives holds.*

i. $V = \{\lambda, t(\lambda)\}$ *for some* $\lambda \in \mathbb{D}^2$*;*

ii. $V = \mathbb{D}^2$*;*

iii. $V = D \cup t(D)$ *for some balanced disc* D *in* $\mathbb{D}^2$ *such that* D^- *meets the set* $\{(z, z) : z \in \mathbb{T}\}$*;*

iv. $V = V_\beta$ *for some* $\beta \in \mathbb{D}$*, where*

$$V_\beta \stackrel{\mathrm{def}}{=} \{(z, w) \in \mathbb{D}^2 : z + w = \beta + \bar{\beta} z w\}; \tag{4}$$

v. $V = \Delta \cup V_\beta$ *for some* $\beta \in \mathbb{D}$*, where* $\Delta = \{(z,z) : z \in \mathbb{D}\}$*;*

vi. $V = V_{m,r}$ *for some* $r \in (0,1)$ *and* $m \in \mathrm{Aut}\ \mathbb{D}$*, where*

$$V_{m,r} \stackrel{\text{def}}{=} \{(z,w) \in \mathbb{D}^2 : H_r(m(z), m(w)) = 0\} \tag{5}$$

and

$$H_r(z,w) \stackrel{\text{def}}{=} 2zw(r(z+w) + 2 - 2r) - (1+r)(z+w)^2 + 2r(z+w). \tag{6}$$

Moreover, the six types of sets V *in* (i) *to* (vi) *are mutually exclusive.*

It is striking that there are three species of set in $\mathbb{D}^2$ that have the symmetric extension property but do not resemble any of the types in Theorem 2.

Applications to the theory of spectral sets

One of motivations for the study of the norm-preserving extension property in a domain of holomorphy is to prove refinements of the inequalities of von Neumann and Andô.

Theorem 8. [Andô inequality] *If $T = (T_1, T_2)$ is a contractive commuting pair of operators on a Hilbert space, then*

$$\|p(T)\| \leq \sup_{\mathbb{D}^2} |p|$$

holds for all p polynomials in two variables.

Here an *operator* means a bounded linear operator on a Hilbert space, and a *contraction* means an operator of norm at most 1.

A *spectral set* for a commuting n-tuple T of operators is a set $V \subseteq \mathbb{C}^n$ such that $\sigma(T) \subseteq V$ and, for every holomorphic function f in a neighborhood of V,

$$\|f(T)\| \leq \sup_V |f|.$$

A-von Neumann sets in $\mathbb{D}^2$

Another formulation of Ando's inequality is that $\mathbb{D}^2$ is a spectral set for any commuting pair of contractions whose joint spectrum is contained in $\mathbb{D}^2$. Isolating the role of $\mathbb{D}^2$ in this statement and generalizing it to arbitrary subsets of $H^\infty(V)$, Agler and McCarthy introduced the following notion.

Definition 4. *Let $V \subseteq \mathbb{C}^2$ and let $A \subseteq H^\infty(V)$. Then V is an A*-von Neumann set *if the inequality*

$$\|f(T)\| \leq \sup_V |f|$$

holds for all $f \in A$ and all pairs T of commuting contractions which are subordinate to V.

Here $H^\infty(V)$ is the algebra of functions $f|V$ where f is bounded on V and holomorphic in some neighborhood U_f of V.

The subordination is the natural notion that ensures that the operator $f(T)$ be well defined.

An A-spectral set for T

Definition 5. *Let $V \subseteq \mathbb{C}^n$, let $A \subseteq H^\infty(V)$ and let T be an n-tuple of commuting operators. V is an A-*spectral set* for T if T is subordinate to V and, for every $f \in A$,*

$$\|f(T)\| \leq \sup_V |f|. \tag{7}$$

Thus V is an A-von Neumann set if V is an A-spectral set for every pair T of commuting contractions which is subordinate to V.

T **is** $\boldsymbol{subordinate}$ **to** V

Definition 6. *Let V be a subset of $\mathbb{C}^n$ and T be an n-tuple of commuting operators on a Hilbert space. T is* subordinate *to V if the spectrum $\sigma(T)$ is a subset of V and every holomorphic function on a neighborhood of V that vanishes on V annihilates T.*

Clearly, if T is subordinate to V and g is the restriction to V of a holomorphic function f on a neighborhood of V then we may uniquely define $g(T)$ to be $f(T)$, where $f(T)$ is defined by the Taylor functional calculus. Thus, if T is subordinate to V then the map $g \mapsto g(T)$ is a functional calculus for $H^\infty(V)$.

Symmetric algebraic sets in $\mathbb{D}^2$ which are $H^\infty_{\mathrm{sym}}(V)$-von Neumann sets

Theorem 9. [Agler and McCarthy] *Let $V \subseteq \mathbb{D}^2$ and let $A \subseteq H^\infty(V)$, then V is an A-von Neumann set $\iff$ V has the A-extension property relative to $\mathbb{D}^2$.*

Therefore, Theorem 7 enables us to give an explicit description of the $H^\infty_{\mathrm{sym}}(V)$-von Neumann sets in $\mathbb{D}^2$.

Theorem 10. *Let V be a symmetric algebraic set in $\mathbb{D}^2$. Then V is an $H^\infty_{\mathrm{sym}}(V)$-von Neumann set $\iff$ V has one of the the six forms* (i) *to* (vi) *in Theorem 7.*

Theorem 10 states that the sets V of Theorem 7 are the only symmetric algebraic sets of $\mathbb{D}^2$ for which the inequality

$$\|f(T)\| \leq \sup_V |f|$$

holds for all bounded symmetric holomorphic functions f on V and all pairs of commuting contractions T subordinate to V.

(G, A)-von Neumann sets are sets with A-extension property relative to G

The A-von Neumann sets of Definition 4 are very much tied to the bidisc. One can define a similar notion for other subsets of $\mathbb{C}^2$. Let us illustrate with the symmetrized bidisc.

Definition 7. *A pair T of commuting bounded linear operators is a Γ-*contraction *if Γ is a spectral set for T.*
Let $V \subseteq G$ and let $A \subseteq H^\infty(V)$. Then V is a (G, A)-von Neumann set *if V is an A-spectral set for every Γ-contraction T subordinate to V.*

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Theorem 11. *Let $V \subseteq G$ and let $A \subseteq H^\infty(V)$. Then V is a (G, A)-von Neumann set if and only if V has the A-extension property relative to G.*

(G, A)-von Neumann sets

In the event that $A = H^\infty(V)$ for a subset V of G, we can describe all (G, A)-von Neumann sets.

Theorem 12. *Let V be an algebraic subset of G. Then V is a $(G, H^\infty(V))$-von Neumann set in G if and only if either V is a retract in G or $V = \mathcal{R} \cup \mathcal{D}$ for some flat geodesic $\mathcal{D}$ in G.*

Theorem 12 states that the described sets V are the only algebraic sets of G for which the inequality

$$\|f(T)\| \leq \sup_V |f|$$

holds for all bounded holomorphic functions $f \in H^\infty(V)$ and all Γ-contraction T subordinate to V.

Anomalous sets with the norm-preserving extension property in some other domains

We observe that in any domain which contains G as a holomorphic retract there are sets that have the norm-preserving extension property but are not retracts.

In particular this observation applies to the 2×2 spectral ball (which comprises the 2×2 matrices of spectral radius less than one) and two domains in $\mathbb{C}^3$ known as the tetrablock and the pentablock.

References

[1] J. Agler and J. E. McCarthy. Norm preserving extensions of holomorphic functions from subvarieties of the bidisk. *Ann. of Math.*, 157(1): 289–312, 2003.

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Thank you