Bloch functions and asymptotic tail variance

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The Bergman projection

We let $\mathbb{D}$ be the open unit disk.

Let $\mathbf{P}$ denote the Bergman projection

$$\mathbf{P}f(z) := \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} \, \mathrm{d}A(w), \qquad z \in \mathbb{D},$$

which is well-defined if $f \in L^1(\mathbb{D})$. It is well-known that $\mathbf{P}$ maps $L^p(\mathbb{D}) \to L^p(\mathbb{D})$ for $1 < p < +\infty$, and that

$$\mathbf{P} : L^2(\mathbb{D}) \to L^2(\mathbb{D})$$

is a norm contraction. We write $\langle \cdot, \cdot \rangle_{\mathbb{D}}$ for the sesquilinear form

$$\langle f, g \rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(z) \bar{g}(z) \mathrm{d}A(z),$$

which is well-defined if $f\bar{g} \in L^1(\mathbb{D})$. We shall be concerned with the space $\mathbf{P}L^\infty(\mathbb{D})$, supplied with the canonical norm

$$\|f\|_{\mathbf{P}L^\infty(\mathbb{D})} := \inf \left\{ \|\mu\|_{L^\infty(\mathbb{D})} : \mu \in L^\infty(\mathbb{D}) \quad \text{and} \quad f = \mathbf{P}\mu \right\}.$$

The Bloch space

It is well-known that as a space, $\mathbf{P}L^\infty(\mathbb{D}) = \mathcal{B}(\mathbb{D})$, the *Bloch space*. This seems to have been observed first in a 1976 paper by Coifman, Rochberg, Weiss [CRW]. We recall that the Bloch space consists of all holomorphic $f : \mathbb{D} \to \mathbb{C}$ subject to the seminorm boundedness condition

$$\|f\|_{\mathcal{B}(\mathbb{D})} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < +\infty.$$

Indeed, if $f = \mathbf{P}\mu$, where $\|\mu\|_{L^\infty(\mathbb{D})} \le 1$, then

$$\begin{aligned}(1-|z|^2)|(\mathbf{P}\mu)'(z)| &= 2(1-|z|^2)\left|\int_{\mathbb{D}} \frac{\bar{w}\mu(w)}{(1-z\bar{w})^3}\mathrm{d}A(w)\right| \\ \le 2(1-|z|^2)\int_{\mathbb{D}}\frac{|w|}{|1-z\bar{w}|^3}\mathrm{d}A(w) &= 2(1-|z|^2)\sum_{j=0}^{+\infty}\frac{[(\frac{3}{2})_j]^2}{(j!)^2(j+\frac{3}{2})}|z|^{2j} \le \frac{8}{\pi},\end{aligned} \tag{1}$$

where the main loss of information is in the application of the triangle inequality.

The Bloch space, cont

On the other hand, if $f \in \mathcal{B}(\mathbb{D})$ with $f(0) = f'(0) = 0$, and we put

$$\mu_f(w) := (1 - |w|^2)\frac{f'(w)}{\bar{w}}, \qquad w \in \mathbb{D},$$

then $\mu_f \in L^\infty(\mathbb{D})$, and

$$|\mu_f(w)| = (1 - |w|^2)\left|\frac{f'(w)}{w}\right| \le (1 + \mathrm{o}(1))\|f\|_{\mathcal{B}(\mathbb{D})} \quad \text{as} \quad |w| \to 1,$$

while

$$\mathbf{P}\mu_f(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)f'(w)}{\bar{w}(1 - z\bar{w})^2}\mathrm{d}A(w) = f(z) - f(0) = f(z), \qquad z \in \mathbb{D}.$$

so essentially there would appear to be a gap of the size $8/\pi$ between the two norms. Perälä (Per) has shown that the bound $8/\pi$ in (1) is best possible.

The Bloch space as a dual space

It is known that with respect to $\langle \cdot, \cdot \rangle_{\mathbb{D}}$, $\mathcal{B}(\mathbb{D})$ can be identified with the dual space of the Bergman space $A^1(\mathbb{D})$ in much the same way that $\mathrm{BMOA}(\mathbb{D})$ is the dual space of $H^1(\mathbb{D})$ with respect to the dual action on the circle $\mathbb{T}$:

$$\langle f, g \rangle_{\mathbb{T}} := \int_{\mathbb{T}} f(z)\bar{g}(z)\mathrm{d}s(z),$$

where $\mathrm{d}s(z) := |\mathrm{d}z|/(2\pi)$ is normalized arc length measure. Indeed, the norm induced on $\mathcal{B}(\mathbb{D})$ by $A^1(\mathbb{D})$ is that of $\mathbf{P}L^\infty(\mathbb{D})$:

Proposition 1

For $f \in \mathcal{B}(\mathbb{D})$, we have that

$$\|f\|_{\mathbf{P}L^\infty(\mathbb{D})} = \sup\left\{ |\langle f, g \rangle_{\mathbb{D}}| : \ g \in A^2(\mathbb{D}), \ \|g\|_{A^1(\mathbb{D})} \le 1 \right\}.$$

Proof of Proposition 1

If $f = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, then

$$\langle f, g\rangle_{\mathbb{D}} = \langle \mathbf{P}\mu, g\rangle_{\mathbb{D}} = \langle \mu, g\rangle_{\mathbb{D}}, \qquad g \in A^2(\mathbb{D}),$$

so that

$$|\langle f, g\rangle_{\mathbb{D}}| = |\langle \mu, g\rangle_{\mathbb{D}}| \le \|\mu\|_{L^\infty(\mathbb{D})}\|g\|_{A^1(\mathbb{D})}, \qquad g \in A^2(\mathbb{D}).$$

It is now immediate that

$$\sup\left\{|\langle f, g\rangle_{\mathbb{D}}| : \; g \in A^2(\mathbb{D}),\;\; \|g\|_{A^1(\mathbb{D})} \le 1\right\} \le \|f\|_{\mathbf{P}L^\infty(\mathbb{D})}.$$

On the other hand, if, for $f \in \mathcal{B}(\mathbb{D})$, we have that

$$\sup\left\{|\langle f, g\rangle_{\mathbb{D}}| : \; g \in A^2(\mathbb{D}),\;\; \|g\|_{A^1(\mathbb{D})} \le 1\right\} = M,$$

then $g \mapsto \langle g, f\rangle_{\mathbb{D}}$ defines a linear functional on $A^1(\mathbb{D})$ of norm M, which by the Hahn-Banach theorem has an extension to a linear functional on $L^1(\mathbb{D})$ which also has norm M. The extended linear functional is then represented by an element $\mu \in L^\infty(\mathbb{D})$, with $\|\mu\|_{L^\infty(\mathbb{D})} = M$:

$$\langle g, f\rangle_{\mathbb{D}} = \langle g, \mu\rangle_{\mathbb{D}}, \qquad g \in A^2(\mathbb{D}),$$

and it is clear that $f = \mathbf{P}\mu$. By the definition of the norm on $\mathbf{P}L^\infty(\mathbb{D})$, then,

$$\|f\|_{\mathbf{P}L^\infty(\mathbb{D})} \le \|\mu\|_{L^\infty(\mathbb{D})} = M = \sup\left\{|\langle f, g\rangle_{\mathbb{D}}| : \; g \in A^2(\mathbb{D}),\;\; \|g\|_{A^1(\mathbb{D})} \le 1\right\},$$

and we have arrived at the reverse inequality. The proof is complete.

Duality and dilations

For a function f, we let $f_r(z) := f(rz)$ denote its dilate. We shall need the following identity.

Proposition 2

Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and that $h \in L^\infty(\mathbb{T})$ is extended harmonically to the interior $\mathbb{D}$. Then

$$\langle zg_r, h\rangle_{\mathbb{T}} = \langle g_r, \partial h\rangle_{\mathbb{D}} = \langle g, (\partial h)_r\rangle_{\mathbb{D}} = \langle \mathbf{P}\mu, (\partial h)_r\rangle_{\mathbb{D}} = \langle \mu, (\partial h)_r\rangle_{\mathbb{D}}.$$

Proof of Proposition 2

The first equality follows from Green's formula. The second step uses that the dilation is self-adjoint, which is easy to check using Taylor series expansions. The third equality expresses that $g = \mathbf{P}\mu$, while the fourth uses that $\mathbf{P}$ is self-adjoint and preserves the holomorphic functions.

Basic estimates

Corollary 3

Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and that $h \in L^\infty(\mathbb{T})$ is extended harmonically to the interior $\mathbb{D}$. Then

$$|\langle z g_r, h\rangle_{\mathbb{T}}| \le \|\mu\|_{L^\infty(\mathbb{D})}\|(\partial h)_r\|_{A^1(\mathbb{D})}.$$

For positive η and $0 < r < 1$, let $\mathcal{E}_g(r,\eta)$ denote the set

$$\mathcal{E}_g(r,\eta) := \{\zeta \in \mathbb{T} : \operatorname{Re}(\zeta g(r\zeta)) \ge \eta\}.$$

Corollary 4

If we put

$$h(\zeta) := \frac{1_{\mathcal{E}_g(r,\eta)}}{|\mathcal{E}_g(r,\eta)|_s},$$

extended harmonically to the interior, then

$$\eta \le \|\mu\|_{L^\infty(\mathbb{D})}\|(\partial h)_r\|_{A^1(\mathbb{D})}.$$

The norm of the dilate in $A^1(\mathbb{D})$

We will assume, without loss of generality, that $\|\mu\|_{L^\infty(\mathbb{D})} = 1$. We will need to estimate $\|(\partial h)_r\|_{A^1(\mathbb{D})}$, provided $h \geq 0$ with $h(0) = 1$ (so that h defines a probability density on the circle $\mathbb{T}$).

From the Cauchy-Schwarz inequality we know that

$$\int_{\mathbb{D}} |(\partial h)(rz)| \mathrm{d}A(z) = \frac{1}{r^2} \int_{\mathbb{D}(0,r)} |\partial h(z)| \mathrm{d}A(z)$$

$$\leq \frac{1}{r^2} \left\{ \int_{\mathbb{D}(0,r)} \frac{h(z) \mathrm{d}A(z)}{1 - |z|^2} \right\}^{1/2} \left\{ \int_{\mathbb{D}(0,r)} \frac{|\partial h(z)|^2}{h(z)} (1 - |z|^2) \mathrm{d}A(z) \right\}^{1/2}$$

$$= \frac{h(0)^{1/2}}{r^2} \left\{ \int_{\mathbb{D}(0,r)} \frac{\mathrm{d}A(z)}{1 - |z|^2} \right\}^{1/2} \left\{ \int_{\mathbb{D}(0,r)} \frac{|\partial h(z)|^2}{h(z)} (1 - |z|^2) \mathrm{d}A(z) \right\}^{1/2}$$

$$= \frac{1}{r^2} \left\{ \log \frac{1}{1 - r^2} \right\}^{1/2} \left\{ \int_{\mathbb{D}(0,r)} \frac{|\partial h(z)|^2}{h(z)} (1 - |z|^2) \mathrm{d}A(z) \right\}^{1/2}. \quad (2)$$

So, it remains to control the last integral in (2).

The key estimate

KEY ESTIMATE (Anentropy bound)

For $h \geq 0$ bounded and harmonic in $\mathbb{D}$ with $h(0) = 1$, we have that

$$\int_{\mathbb{D}(0,r)} \frac{|\partial h(z)|^2}{h(z)} (1 - |z|^2) \mathrm{d}A(z) \leq \int_{\mathbb{T}} h \log h \, \mathrm{d}s.$$

Remark

This estimate is very sharp. We will not explain the proof of this bound here.

Consequence of the key estimate

Corollary 5

If h is as in Corollary 4, and $\|\mu\|_{L^\infty(\mathbb{D})} = 1$, then

$$\eta \leq \frac{1}{r^2}\left\{\log\frac{1}{1-r^2}\right\}^{1/2}\left\{\int_{\mathbb{T}} h\log h\,\mathrm{d}s\right\}^{1/2}$$
$$= \frac{1}{r^2}\left\{\log\frac{1}{1-r^2}\right\}^{1/2}\left\{\log\frac{1}{|\mathcal{E}_g(r,\eta)|_s}\right\}^{1/2}.$$

Remark

(Weak type bound) In other words, we obtain an estimate of the length of the set

$$\mathcal{E}_g(r,\eta) = \{\zeta \in \mathbb{T} : \operatorname{Re}(\zeta g(r\zeta)) \geq \eta\},$$

which reads

$$|\mathcal{E}_g(r,\eta)|_s \leq \exp\left\{-\frac{r^4\eta^2}{\log\frac{1}{1-r^2}}\right\}. \tag{3}$$

This is *Gaussian tail behavior*.

From weak to strong type bound

We can turn the weak type bound into a strong type bound, at a small cost.

MAIN THEOREM

Suppose $g = \mathbf{P}\mu$ where $\|\mu\|_{L^\infty(\mathbb{D})} = 1$, and let

$$I_g(a, r) := \int_{\mathbb{T}} \exp\left\{a \frac{r^4|g(r\zeta)|^2}{\log \frac{1}{1-r^2}}\right\} \mathrm{d}s(\zeta),$$

for $a \geq 0$ and $0 < r < 1$.

(a) If $0 \leq a < 1$, we have that $I_g(a, r) \leq C(a)$ independently of μ and r, where $C(a) := 10(1-a)^{-3/2}$.

(b) If $a > 1$, there exists μ_0 with $\|\mu_0\|_{L^\infty(\mathbb{D})} = 1$ such that with $g_0 := \mathbf{P}\mu_0$, $I_{g_0}(a, r) \to +\infty$ as $r \to 1^-$.

Remark

For $0 < a < \frac{\pi^2}{64} = 0.154\ldots$, the bound (with another constant) follows from an estimate of Makarov (see [Mak] and [Pombk], p. 186, p. 188).

The case (b) of the main theorem

The function μ_0 is explicit,

$$\mu_0(z) := \frac{1-\bar{z}}{1-z},$$

so that its Bergman projection may be calculated:

$$\mathbf{P}\mu_0(z) = \frac{1}{z^2}\log\frac{1}{1-z} - \frac{1}{z}.$$

The assertion is now just a matter of direct verification.

Interpretation as asymptotic tail variance

Consider

$$X_r(\zeta) := \frac{r^2 g(r\zeta)}{\log \frac{1}{1-r^2}}, \qquad \zeta \in \mathbb{T},$$

as a random variable which is an almost rotationally-invariant complex Gaussian. This allows us to define the *asymptotic tail variance*:

$$\operatorname{atvar} g := \inf \left\{ \tau > 0 : \limsup_{r \to 1^-} \mathbb{E}\, \mathrm{e}^{|X_r|^2/\tau} < +\infty \right\}$$

This tail variance should be compared with the asymptotic variance of McMullen [McM].

Remark

Similar tail variances can be defined for individual probability distributions on the plane $\mathbb{C}$ or on the line $\mathbb{R}$.

Consequences for the integral means spectrum of conformal mappings with quasiconformal extension

We recall that $B(k,t)$ is the universal integral means spectrum for conformal mappings of the exterior disk preserving the point at infinity, having a k-quasiconfrmal extension to the whole plane.

MAIN COROLLARY

We have that

$$B(k,t) \leq \begin{cases} \frac{1}{4}k^2|t|^2(1+7k)^2, & |t| \leq \frac{2}{k(1+7k)^2}, \\ k|t| - \frac{1}{(1+7k)^2}, & |t| \geq \frac{2}{k(1+7k)^2}. \end{cases}$$

Remark

Prause and Smirnov [PS] conjecture that this estimate is an equality, if the expression $(1+7k)^2$ everywhere gets replaced by 1. For small k, we are therefore very close to the conjectured value.

Application to the dimension of quasicircles

To test the power of the main corollary, we observe that together with Pommerenke's ([Pombk], p. 241) dimension formula, and a symmetrization procedure which apparently originates with Kühnau. Let $D(k)$ denote the maximal (Minkowski) dimension of the image of the unit circle under a k-quasiconformal mapping.

SECONDARY COROLLARY

We have that

$$D(k) \leq 1 + k^2 + O(k^3).$$

Remark

This should be compared with Smirnov's theorem $D(k) \leq 1 + k^2$ (see [Sm]).

Bibliography

[AIPP] Astala, K., Ivrii, O., Perälä, A., Prause, I., *Asymptotic variance of the Beurling transform*. Geom. Funct. Anal., to appear.

[AIM] Astala, K., Iwaniec, T., Martin, G., *Elliptic partial differential equations and quasiconformal mappings in the plane*. Princeton Mathematical Series, **48**. Princeton University Press, Princeton, NJ, 2009.

[HKZ] Hedenmalm, H., Korenblum, B., Zhu, K., *Theory of Bergman spaces*. Graduate Texts in Mathematics, **199**. Springer-Verlag, New York, 2000.

[HS1] Hedenmalm, H., Shimorin, S., *Weighted Bergman spaces and the integral means spectrum of conformal mappings*. Duke Math. J. **127** (2005), no. 2, 341-393.

Bibliography, cont

[CRW] Coifman, R. R., Rochberg, R., Weiss, G., *Factorization theorems for Hardy spaces in several variables*. Ann. of Math. **103** (1976), 611-635.

[HS2] Hedenmalm, H., Shimorin, S., *On the universal integral means spectrum of conformal mappings near the origin*. Proc. Amer. Math. Soc. **135** (2007), no. 7, 2249-2255

[K] Korenblum, B., *BMO estimates and radial growth of Bloch functions*. Bull. Amer. Math. Soc. **12** (1985), no. 1, 99-102.

[Mak] Makarov, N. G., *On the distortion of boundary sets under conformal mapping*. Proc. Lond. Math. Soc. **51** (1985), 369-384.

[McM] McMullen, C. T., *Thermodynamics, dimension, and the Weil-Petersson metric*. Invent. Math. **173** (2008), 365-425.

[Per] Perälä, A., *On the optimal constant for the Bergman projection onto the Bloch space*. Ann. Acad. Sci. Fenn. Math. **37** (2012), no. 1, 245-249.

Bibliography, cont

[Pombk] Pommerenke, Ch., *Boundary behaviour of conformal maps*. Grundlehren 299, Springer, 1992.

[PS] Prause, I., Smirnov, S., *Quasisymmetric distortion spectrum*. Bull. Lond. Math. Soc. **43** (2011), no. 2, 267-277.

[Sm] Smirnov, S., *Dimension of quasicircles*. Acta Math. **205** (2010), no. 1, 189-197.