Holomorphic self-maps of the unit disc with two fixed points

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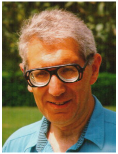

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Let $\mathfrak{P}$ be the set of all functions f holomorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon \ |z| < 1\}$ and taking values in $\mathbb{D}$, that is, $f(\mathbb{D}) \subset \mathbb{D}$.

The aim of the present note is to study the case when f in $\mathfrak{P}$ has two fixed points.

We point out first that if f belongs to $\mathfrak{P}$ and $f(z) \not\equiv z$, then by the hyperbolic metric principle there can be at most one fixed point in $\mathbb{D}$. On the other hand, a function f in $\mathfrak{P}$ may have fixed points at the boundary $\partial\mathbb{D}$ of the unit disc $\mathbb{D}$. We recall that a point $\varkappa \in \partial\mathbb{D}$ (that is, $|\varkappa| = 1$) is said to be a fixed point for $f \in \mathfrak{P}$ if

$$\lim_{r\to 1} f(r\varkappa) = \varkappa.$$

It is a remarkable fact that the angular derivative

$$f'(\varkappa) = \angle \lim_{z \to \varkappa} \frac{f(z) - \varkappa}{z - \varkappa}$$

at a boundary fixed point $\varkappa$ always exists and $f'(\varkappa)$ is equal to either $+\infty$ or a positive real number.

If it is finite, $f'(z)$ has the same angular limit. The symbol $\angle \lim_{z \to \varkappa}$ means that z stays within an angle at $\varkappa$ less than π, and the limit is referred to as an angular limit.

For convenience, we have normalized the function $f \in \mathfrak{P}$ so that its fixed points are 0 and 1 or -1 and 1.

In this connection we single out the class $\mathfrak{P}[0, 1]$ of mappings $f : \mathbb{D} \mapsto \mathbb{D}$ that fix both the origin (that is, $f(0) = 0$) and the boundary point $z = 1$ in the sense of the angular limit and have a finite angular derivative at $z = 1$.

Analogously, we denote the class of functions f in $\mathfrak{P}$ that fix points $z = \pm 1$ and have finite angular derivatives $f'(-1)$, $f'(1)$ by $\mathfrak{P}[-1, 1]$.

It is a consequence of the Julia-Carathéodory theorem that $f'(1) \geq 1$ for any function f in $\mathfrak{P}[0,1]$. Moreover, the equality $f'(1) = 1$ is possible only if $f(z) \equiv z$.

Note that if f in $\mathfrak{P}[0,1]$ is univalent and $f'(1) = \alpha > 1$ then

$$|f'(0)| \geq 1/\alpha^2.$$

This result has been obtained by many authors using a variety methods (Solynin, Pommerenke and Vasil'ev, Dubinin and Kim, Anderson and Vasil'ev).

The following result gives detailed information on the values $f'(0), f''(0), f'(1)$ in the class $\mathfrak{P}[0,1]$.

Theorem (1)

Let $f(z) = c_1 z + c_2 z^2 + \dots$ be in $\mathfrak{P}[0;1]$ and $f'(1) = \alpha > 1$. Then

$$\left| c_1 - \frac{1}{\alpha} \right| \leq \frac{\alpha - 1}{\alpha}, \qquad \left| c_2 - \frac{(c_1 - 1)^2}{\alpha - 1} \right| \leq 1 - \frac{|\alpha c_1 - 1|^2}{(\alpha - 1)^2},$$

and $c_1 \neq 1$.

Note that the inequality

$$\left| f'(0) - \frac{1}{\alpha} \right| \leq \frac{\alpha - 1}{\alpha}$$

implies for $\alpha \in (1,2)$ the following

$$|f'(0)| \geq (2 - \alpha)/\alpha.$$

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Thus, the value $f'(1) = 2$ is critical.

The result of the following theorem is more informative.

Theorem (2)

Let f be in $\mathfrak{P}[0,1]$ and $f'(1) = \alpha$, $1 < \alpha < 2$. Then f is univalent in the domain

$$A_M = \{z \in \mathbb{D}\colon\ |1 - z| < M(1 - |z|)\},$$

where $M = 1/\sqrt{\alpha - 1}$.

Note that A_M contains the disc

$$|z| < \frac{1 - \sqrt{\alpha - 1}}{1 + \sqrt{\alpha - 1}}.$$

Moreover, for $\alpha \in (1,2)$ the function

$$f(z) \;=\; z\frac{\alpha z + (2-\alpha)}{\alpha + (2-\alpha)z}$$

belongs to $\mathfrak{P}[0,1]$ and $f'(1) = \alpha$. Also, the point

$$z_\alpha \;=\; -\frac{1-\sqrt{\alpha-1}}{1+\sqrt{\alpha-1}}$$

lies on ∂A_M, $M = 1/\sqrt{\alpha-1}$, and $f'(z_\alpha) = 0$.

This implies that the parameter M of the domain A_M in Theorem 2 is the best possible.

A description of class $\mathfrak{P}[0,1]$ can be obtained by using a result, which is due to R. Nevanlinna.

A function f holomorphic in $\mathbb{D}$ belongs to $\mathfrak{P}[0,1]$ if and only if the function $h_f(z) = (1+f(z))/(1-f(z))$ admits a representation

$$h_f(z) \;=\; \lambda\frac{1+z}{1-z} + (1-\lambda)\int\limits_{\mathbb{T}} \frac{1+\varkappa z}{1-\varkappa z}d\mu(\varkappa)$$

for some probability measure μ on the unit circle $\mathbb{T} = \partial\mathbb{D}$ with $\mu(\{1\}) = 0$, and $\lambda \in (0,1)$.

Note that the measure

$$\widetilde{\mu} \;=\; \lambda\delta_1 + (1-\lambda)\mu$$

is so-called Aleksandrov–Clark measure of f at the point $\zeta = 1$.

Now we consider the case when f fixes two boundary points $z = 1$ and $z = -1$.

Suppose that f belongs to $\mathfrak{P}[-1, 1]$. By the Julia-Carathéodory theorem we have the inequality $f'(-1)f'(1) \geq 1$. Moreover, the equality $f'(-1)f'(1) = 1$ is possible only if f is a linear fractional transformation of the unit disc onto itself.

The following theorem gives a description of the class $\mathfrak{P}[-1, 1]$.

Theorem (3)

Let f be in $\mathfrak{P}[-1;1]$ and $f'(-1)=\varrho$, $f'(-1)f'(1)=\alpha>1$. Then the function $h(z)=(1+f(z))/(1-f(z))$ admits a representation in the form

$$h(z) \;=\; \varrho\left\{\lambda\frac{1+z}{1-z}+(1-\lambda)\frac{1+z}{2}\int\limits_{\mathbb{T}}\frac{1+\varkappa}{1-\varkappa z}d\mu(\varkappa)\right\} \qquad (1)$$

where $\lambda=1/\alpha$ and μ is a probability measure on $\mathbb{T}$ with $\mu(\{-1,1\})=0$. Conversely, if $\varrho>0$, $0<\lambda<1$, and μ is a probability measure on $\mathbb{T}$ with $\mu(\{-1,1\})=0$, then the function $h(z)$ in (1) is holomorphic in $\mathbb{D}$ with positive real part, and $f(z)=(h(z)-1)/(h(z)+1)$ belongs to $\mathfrak{P}[-1,1]$ with $f'(-1)=\varrho$, $f'(-1)f'(1)=1/(\lambda\varrho)$.

Remark

A positive Borel measure ν on $\mathbb{T}$ is an Aleksandrov–Clark measure of a function f in $\mathfrak{P}[-1;1]$ with $f'(-1) = \varrho$, $f'(-1)f'(1) = \alpha > 1$ at the point 1, if and only if it admits a representation

$$\nu \;=\; \frac{\varrho}{\alpha}\delta_1 + \frac{\varrho(\alpha-1)}{\alpha}\widetilde{\mu},$$

where

$$d\widetilde{\mu}(\varkappa) \;=\; \frac{1}{2}(1+\operatorname{Re}\varkappa)d\mu(\varkappa), \qquad\qquad \varkappa \in \mathbb{T},$$

and μ is a probability measures on $\mathbb{T}$ with $\mu(\{-1,1\}) = 0$.

For $\alpha, \beta > 0$ that satisfy the inequality $\alpha\beta > 1$ we consider the set

$$\mathcal{D}(\alpha,\beta) = \{(f(0), f'(0)) \colon \; f \in \mathfrak{P}[-1,1], f'(1) = \alpha, f'(-1) = \beta\}.$$

Theorem (4)

Let $\alpha, \beta > 0$ and $\alpha\beta > 1$. Then a necessary and sufficient condition for a point (w, ζ) in $\mathbb{C}^2$ to lie in the set $\mathcal{D}(\alpha, \beta)$ is that

$$|w - \theta| \le \tau, \qquad w \ne \theta \pm \tau,$$

$$\left|\zeta - \frac{\alpha+\beta+2}{\alpha\beta-1}\left(w - \frac{\beta-\alpha}{\alpha+\beta+2}\right)^2 - \frac{4}{\alpha+\beta+2}\right|$$

$$\le \frac{1}{\tau}(\tau^2 - |w-\theta|^2),$$

$$\theta = \frac{\beta-\alpha}{(\alpha+1)(\beta+1)}, \qquad \tau = \frac{\alpha\beta-1}{(\alpha+1)(\beta+1)}.$$

Theorem (5)

Let f belongs to $\mathfrak{P}[-1,1]$ and $f'(1)f'(-1) = \alpha$, $1 < \alpha < 2$. Then f is univalent in the domain

$$\Delta_M = \left\{ z \in \mathbb{D} \colon \frac{|1-z^2|}{1-|z|^2} < M \right\},$$

where $M = 1/\sqrt{\alpha - 1}$.

Note that the parameter M of the domain Δ_M in Theorem 5 is the best possible. The boundary $\partial\Delta_M$ is a closed curve consisting of two arcs of circles which pass through $+1$ and -1.

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Thank You!