# 2D Ising model: discrete holomorphicity, orthogonal polynomials and conformal invariance

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[ Sample of a critical 2D Ising configuration (with two disorders), © Clément Hongler (EPFL) ]

XXV Meeting in Mathematical Analysis
Tribute to Victor Havin (1933–2015)
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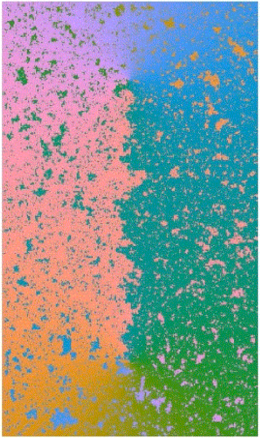
# 2D Ising model: discrete holomorphicity, orthogonal polynomials and conformal invariance

- Nearest-neighbor Ising model in 2D
  - ∘ dimers and Kac–Ward matrices
  - ∘ fermionic observables
  - ∘ discrete holomorphicity at criticality
- Spin correlations via spinor observables
  - ∘ definition of spinor observables
  - ∘ full-plane spinors and formulas for "diagonal" spin-spin expectations in $\mathbb{Z}^2$
- Conformal covariance at criticality
  - ∘ Riemann boundary value problems for holomorphic spinors in continuum
  - ∘ Explicit formulas (CFT prediction)
  - ∘ Convergence (Ch.–Hongler–Izyurov)



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## Nearest-neighbor Ising or Lenz-Ising model in 2D

Definition: *Lenz-Ising model* on a planar graph $G^*$ (dual to $G$) is a random assignment of $+/-$ spins to vertices of $G^*$ (faces of $G$)

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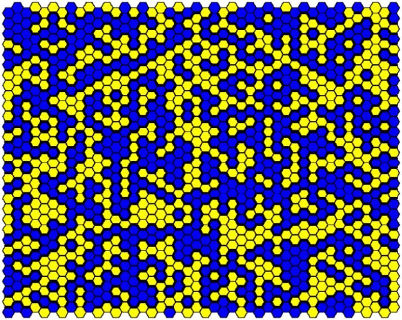

[sample of a honeycomb percolation]

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A: .. according to the following probabilities:

$$\mathbb{P}\left[\mathrm{conf.}\ \sigma \in \{\pm 1\}^{V(G^*)}\right] \;\propto\; \exp\left[\,\beta \sum\nolimits_{e=\langle uv\rangle} J_{uv}\sigma_u\sigma_v\,\right]$$
$$\propto\; \prod\nolimits_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv}\,,$$

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**Disclaimer:**

**no external magnetic field.**

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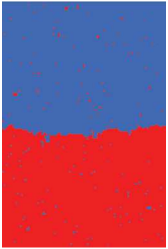
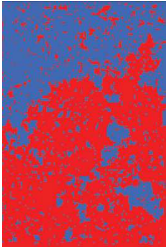
## Phase transition (e.g., on $\mathbb{Z}^2$)

E.g., Dobrushin boundary conditions: $+1$ on $(ab)$ and $-1$ on $(ba)$:



$x < x_{\mathrm{crit}}$



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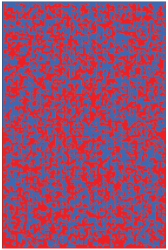

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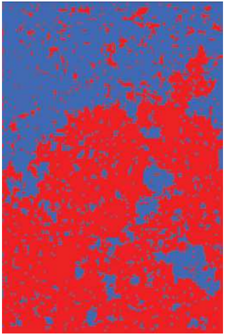
- Ising (1925): no phase transition in 1D $\rightsquigarrow$ doubts about 2+D;
- Peierls (1936): existence of the phase transition in 2D;
- Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan(\frac{1}{2} \cdot \frac{\pi}{4})$;
- Onsager (1944): sharp phase transition at $x_{\text{crit}} = \sqrt{2} - 1$.

**At criticality (e.g., on $\mathbb{Z}^2$):**

- Kaufman-Onsager(1948-49), Yang(1952): scaling exponent $\frac{1}{8}$ for the magnetization (some spin correlations in $\mathbb{Z}^2$ at $x \uparrow x_{\mathrm{crit}}$).
- At criticality, for $\Omega_\delta \to \Omega$ and $u_\delta \to u \in \Omega$, it should be $\mathbb{E}_{\Omega_\delta}[\sigma_{u_\delta}] \asymp \delta^{\frac{1}{8}}$ as $\delta \to 0$.



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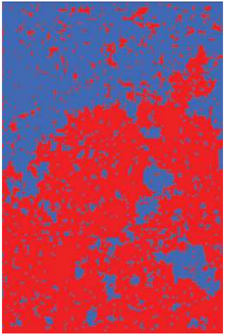

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• **Question for the part #2:** Classical formulas for "diagonal" spin-spin expectations in $\mathbb{Z}^2$ via **spinor observables**

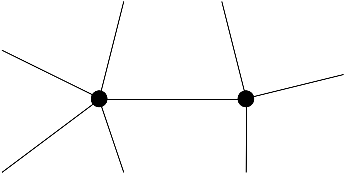
• **Question for the part #3:** Convergence and conformal covariance of spin correlations in **arbitrary planar domains:**

$$\begin{aligned}\delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \dots \sigma_{u_{n,\delta}}] &\to \langle \sigma_{u_1} \dots \sigma_{u_n} \rangle_\Omega \\ &= \langle \sigma_{\varphi(u_1)} \dots \sigma_{\varphi(u_n)} \rangle_{\varphi(\Omega)} \cdot \textstyle\prod_{s=1}^n |\varphi'(u_s)|^{\frac{1}{8}}\end{aligned}$$

## 2D Ising model as a dimer model (on a non-bipartite graph) ( ..., Fisher, Kasteleyn, ..., Kenyon, Dubedat, ... )

• **Partition function** $\mathcal{Z} = \sum_{\sigma \in \{\pm 1\}^{V(G^*)}} \prod_{e=\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv}$

• There exist various representations of the 2D Ising model via dimers on an auxiliary graph



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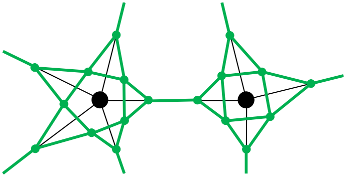

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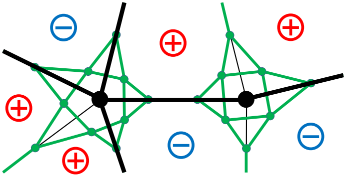

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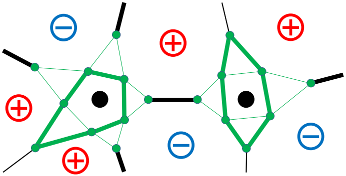

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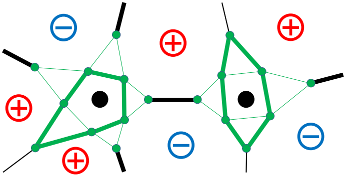

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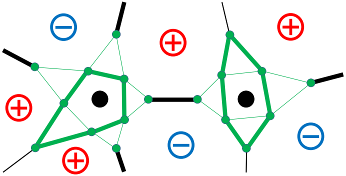

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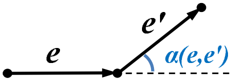
- **Kac–Ward formula (1952–..., 1999–...):** $\mathcal{Z}^2 = \det[\mathrm{Id} - \mathrm{T}]$,

$$\mathrm{T}_{e,e'} = \begin{cases} \exp[\tfrac{i}{2}\alpha(e,e')]\cdot(x_e x_{e'})^{1/2} \\ 0 \end{cases}$$



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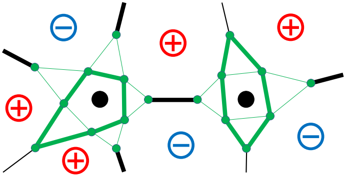

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[ **is equivalent** to the **Kasteleyn theorem for dimers on $G_F$** ]

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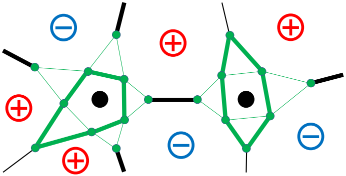

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- Note that $V(G_F) \cong$ $\{$***oriented edges and corners of G***$\}$

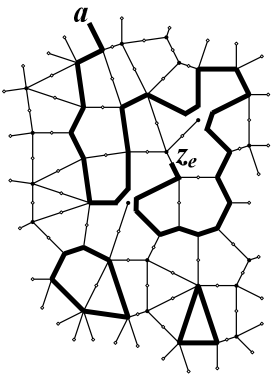
- **Local relations** for the entries $\mathbf{K}^{-1}_{a,e}$ and $\mathbf{K}^{-1}_{a,c}$ of the inverse Kasteleyn matrix: (an equivalent form of) $\mathbf{K}\cdot\mathbf{K}^{-1} = \mathbf{Id}$

## Fermionic observables: combinatorial definition [Smirnov '00s]

For an oriented edge $a$ of $G$ and a midpoint $z_e$ of another edge $e$,

$$F_G(a, z_e) \;:=\; \overline{\eta}_a \sum\nolimits_{\omega \in \mathrm{Conf}_G(a, z_e)} \Big[ e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)} \prod\nolimits_{\langle uv \rangle \in \omega} x_{uv} \Big],$$

where $\eta_a$ denotes the (once and forever fixed) square root of the direction of $a$.



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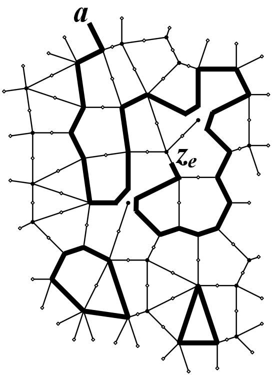


• The factor $e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)}$ does not depend on the way how $\omega$ is split into non-intersecting loops and a path $a \rightsquigarrow z_e$.

• When both $a$ and $e$ are "boundary" edges, the factor $\overline{\eta}_a e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)} = \pm \overline{\eta}_e$ is fixed and $F_G(a, z_e)$ becomes the partition function of the Ising model (on $G^*$) with Dobrushin boundary conditions.

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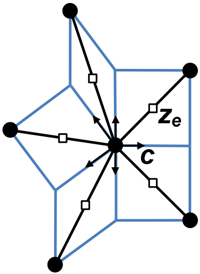

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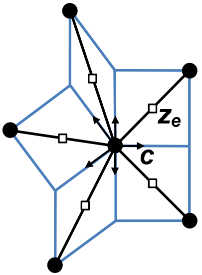
- **S-holomorphicity** (special self-dual weights on isoradial graphs):

$$\boldsymbol{F_G(a, c) \;=\; \mathrm{Proj}[\, F_G(a, z_e)\,;\, \overline{\eta}_c \,]}$$

provided each edge $e$ of $G$ is a diagonal of a rhombic tile with half-angle $\theta_e$ and the Ising model weights are given by $x_e = \tan(\frac{1}{2}\theta_e)$.




- $\Rightarrow$ critical weights on regular grids:
  - square: $x_{\mathrm{crit}} = \tan\frac{\pi}{8} = \sqrt{2} - 1$,
  - honeycomb: $x_{\mathrm{crit}} = \tan\frac{\pi}{6} = 1/\sqrt{3}$, ...

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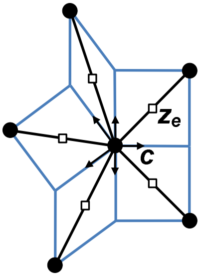


- **Via dimers on $G_F$:** $\quad F_G(a, c) = \overline{\eta}_c \mathrm{K}^{-1}_{c,a}$

$$F_G(a, z_e) = \overline{\eta}_e \mathrm{K}^{-1}_{e,a} + \overline{\eta}_{\overline{e}} \mathrm{K}^{-1}_{\overline{e},a}$$

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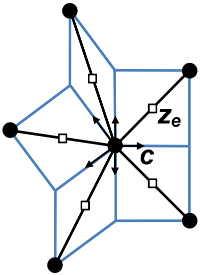


- Fermionic observables *per se* are useful but *do not allow to analyze the spin correlations:* **more involved ones are needed**

## Spinor observables and spin correlations

- spin configurations on $G^*$
  $\leftrightsquigarrow$ domain walls on $G$
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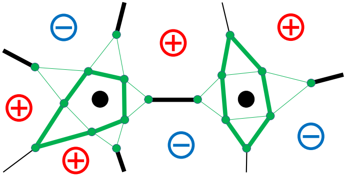

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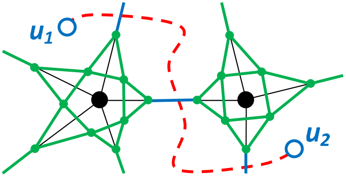
[ $\mathbf{K} = -\mathbf{K}^\top$ is a weighted adjacency matrix of $G_F$ ]

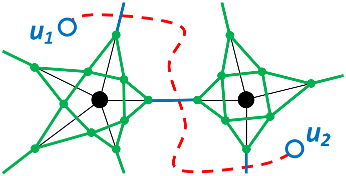



- **Claim**:

$$\mathbb{E}[\sigma_{u_1} \dots \sigma_{u_n}] \;=\; \frac{\mathrm{Pf}[\mathbf{K}_{[u_1,\dots,u_n]}]}{\mathrm{Pf}[K]},$$

where $\mathbf{K}_{[u_1,\dots,u_n]}$ is obtained from $\mathbf{K}$ by changing the sign of its entries on **slits linking** $u_1, \dots, u_n$ (and, possibly, $u_{\mathrm{out}}$) pairwise.

## Spinor observables and spin correlations




- spin configurations on $G^*$
  $\leftrightsquigarrow$ domain walls on $G$
  $\leftrightsquigarrow$ dimers on $G_F$

- **Kasteleyn's theory:** $\mathcal{Z} = \mathrm{Pf}[\,\mathbf{K}\,]$

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- **More invariant way** to think about entries of $\mathbf{K}^{-1}_{[u_1,\ldots,u_n]}$ :

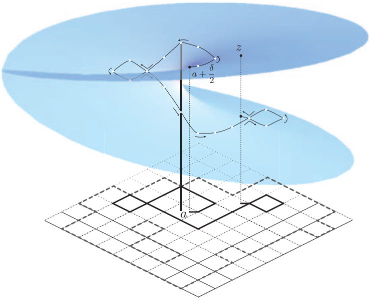
**double-covers of $G$ branching over $u_1, \ldots, u_n$**

## Spinor observables and spin correlations

**Main tool: spinors on the double cover $[\Omega_\delta; u_1, \ldots, u_n]$.**

$$F_{\Omega_\delta}(z) := \left[\mathcal{Z}^{+}_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}]\right]^{-1} \cdot \sum_{\omega \in \mathrm{Conf}_{\Omega_\delta}(u_1^{\rightarrow}, z)} \phi_{u_1,\ldots,u_n}(\omega, z) \cdot x_{\mathrm{crit}}^{\#\mathrm{edges}(\omega)},$$

$$\phi_{u_1,\ldots,u_n}(\omega, z) := e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\omega))} \cdot (-1)^{\#\mathrm{loops}(\omega \setminus \mathrm{p}(\omega))} \cdot \mathrm{sheet}(\mathrm{p}(\omega), z).$$




- $\mathrm{wind}(\mathrm{p}(\gamma))$ is the winding of the path $\mathrm{p}(\gamma) : u_1^{\rightarrow} = u_1 + \frac{\delta}{2} \rightsquigarrow z$;
- $\#\mathrm{loops}$ – those containing an odd number of $u_1, \ldots, u_n$ inside;
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## Spinor observables and spin correlations

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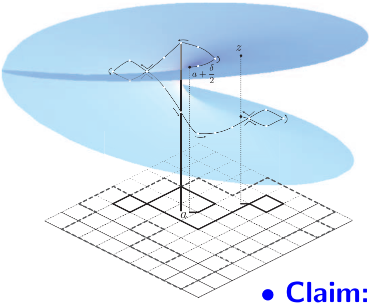


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---

- **Claim:** $F_{\Omega_\delta}(u_1 + \frac{3\delta}{2}) = \dfrac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1+2\delta}\ldots\sigma_{u_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1}\ldots\sigma_{u_n}]}$

## “Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan\frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan\frac{\pi}{8}$ and $D_n(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(0,0)}\sigma_{(2n,0)}]$ where $\mathbb{C}^\diamond = \{(k,s) : k,s \in \mathbb{Z}, k+s \in 2\mathbb{Z}\}$ is the $\frac{\pi}{4}$-rotated $\mathbb{Z}^2$.

**Theorem:** [B.Kaufman–L.Onsager'48-49, C.N.Yang'52]

$\lim_{n\to\infty} D_n(x) = (1-\tan^4\theta)^{\frac{1}{4}} \sim \mathrm{const}\cdot(x_{\mathrm{crit}}-x)^{\frac{1}{4}}$ for $x < x_{\mathrm{crit}}$

[T.T.Wu'66] $D_n(x_{\mathrm{crit}}) = \left(\frac{2}{\pi}\right)^n \prod_{s=1}^{n-1}\left(1-\frac{1}{4s^2}\right)^{s-n} \sim \mathrm{const}\cdot(2n)^{-\frac{1}{4}}$

---

Classical reference for many explicit computations:

B.M. McCoy and T.T. Wu *“The two-dimensional Ising model”*

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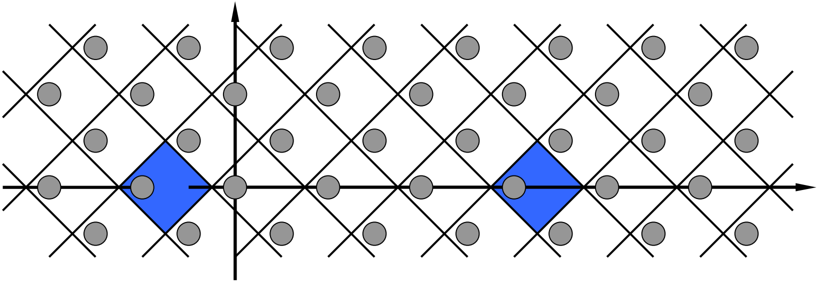
`... we talked to Kakutani and Kakutani talked to Szego, and the mathematicians got there first.`

## “Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan \frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan \frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$



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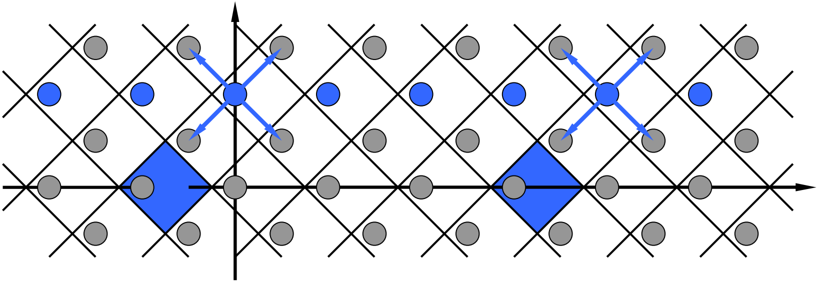

**Local relations** $F_{\mathbb{C}^\diamond}(d) - \frac{m}{4}\sum_{d'\sim d} F_{\mathbb{C}^\diamond}(d') = 0$, $m = \sin(2\theta)$

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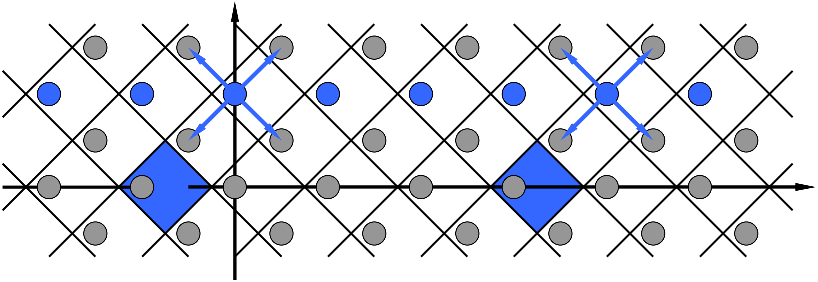

**Local relations** $F_{\mathbb{C}^\diamond}(d) - \frac{m}{4}\sum_{d'\sim d} F_{\mathbb{C}^\diamond}(d') = 0$, $m = \sin(2\theta)$

For $s \geqslant 0$, denote $Q_{n,s}(e^{it}) := D_{n+1} \cdot \sum_{k\in\mathbb{Z}:k+s\in 2\mathbb{Z}} e^{\frac{1}{2}ikt} F_{\mathbb{C}^\diamond}(k,s)$

Then local relations (massive harmonicity) can be rewritten as

$$Q_{n,s}(e^{it}) = (\tfrac{m}{2}\cos\tfrac{t}{2}) \cdot (Q_{n,s-1}(e^{it}) + Q_{n,s+1}(e^{it})), \quad s \geqslant 1.$$

## “Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

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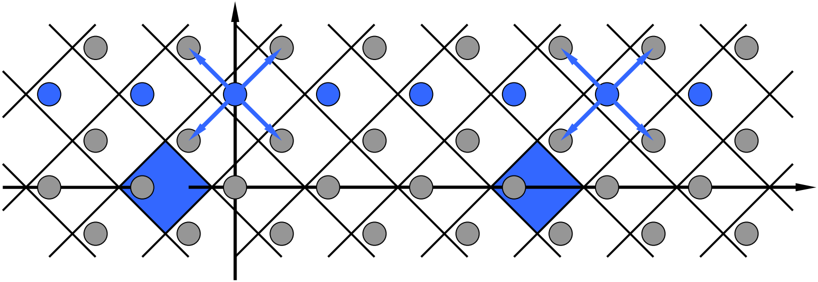

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Boundedness as $s \to \infty \Rightarrow Q_{n,1}(e^{it}) = \left[\frac{1-(1-(m\cos\frac{t}{2})^2)^{\frac{1}{2}}}{m\cos\frac{t}{2}}\right] Q_{n,0}(e^{it})$

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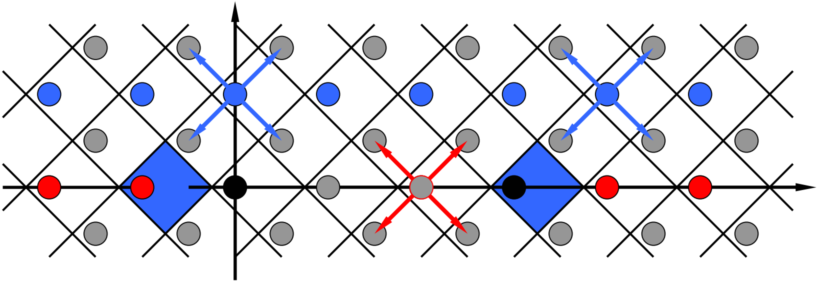

Combinatorics of spinor observables $\Rightarrow$ the following values on $\mathbb{R}$:

$$\begin{aligned} Q_n(e^{it}) := Q_{n,0}(e^{it}) &= \;\; \mathbf{0} \; + \; D_n \; + \ldots + \; D_n^* e^{int} \; + \; \mathbf{0} \\ w(t)Q_n(e^{it}) &= \; \ldots + D_{n+1} + \mathbf{0} + q^2 D_{n+1}^* e^{int} + \ldots\,, \end{aligned}$$

where $\boldsymbol{w(t) = |1 - q^2 e^{it}|}$ and $q := \tan\theta \leqslant 1$.

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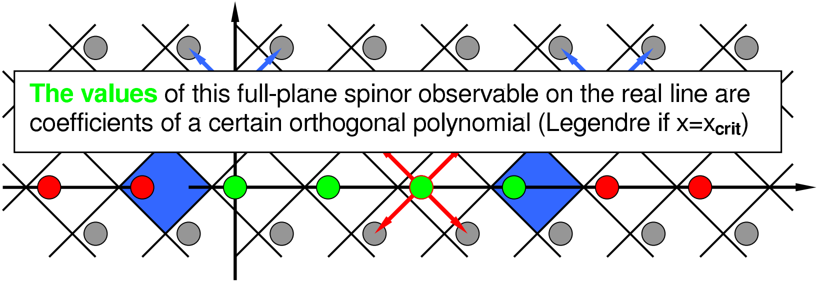


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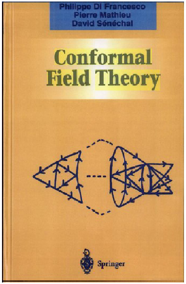
## Conformal covariance of spin correlations at criticality

- Three primary fields:
  $\mathbf{1}$, $\boldsymbol{\sigma}$ (spin), $\varepsilon$ (energy density);
  Scaling exponents: $0$, $\mathbf{\frac{1}{8}}$, $1$.

- **CFT prediction:**
  If $\Omega_\delta \to \Omega$ and $u_{k,\delta} \to u_k$ as $\delta \to 0$, then

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \dots \sigma_{u_{n,\delta}}] \underset{\delta \to 0}{\to} \mathcal{C}_\sigma^n \cdot \langle \boldsymbol{\sigma}_{u_1} \dots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\Omega}$$




where $\mathcal{C}_\sigma$ is a lattice-dependent constant,

$$\langle \sigma_{u_1} \dots \sigma_{u_n} \rangle^{+}_{\Omega} \ = \ \langle \sigma_{\varphi(u_1)} \dots \sigma_{\varphi(u_n)} \rangle^{+}_{\Omega'} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}$$

for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\left[ \langle \boldsymbol{\sigma}_{u_1} \dots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\mathbb{H}} \right]^2 = \prod_{1 \leqslant s \leqslant n} (2 \operatorname{Im} u_s)^{-\frac{1}{4}} \times \sum_{\mu \in \{\pm 1\}^n} \prod_{s<m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\mu_s \mu_m}{2}}$$

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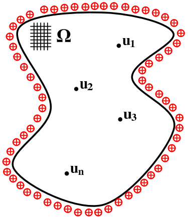


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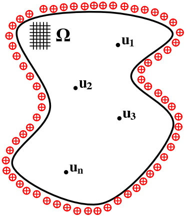


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**General strategy:** • in discrete: encode spatial derivatives as values of discrete holomorphic functions $F^\delta$ that solve some

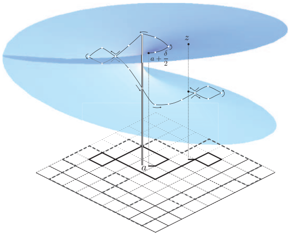
**discrete boundary value problems;**

• discrete→continuum: prove convergence of $F^\delta$ to the solutions $f$ of the similar continuous b.v.p. [non-trivial technicalities];

• continuum→discrete: derive the limit of correlations from the convergence $F^\delta \to f$ [via coefficients at singularities].

# Conformal covariance of spin correlations at criticality

**Example:** to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:



- $f(z^\sharp) \equiv -f(z^\flat)$, branches over $u$;
- $\mathrm{Im}\big[\, f(\zeta)\sqrt{n(\zeta)}\,\big] = 0$ for $\zeta \in \partial\Omega$;
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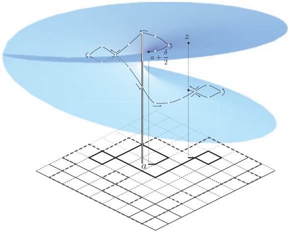

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**Claim:** If $\Omega_\delta$ converges to $\Omega$ as $\delta \to 0$, then

$$\circ \quad (2\delta)^{-1} \log \left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to \mathrm{Re}[\,\mathcal{A}_\Omega(u)\,]\,;$$

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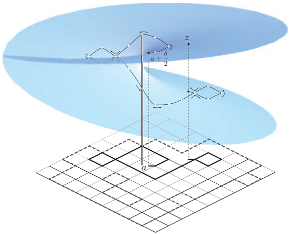


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**Conformal covariance $\frac{1}{8}$ :** for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w))\cdot(\phi'(w))^{1/2}\,;$
- $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z))\cdot\phi'(z) + \frac{1}{8}\cdot\phi''(z)/\phi'(z)\,.$

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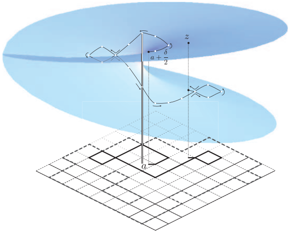

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**Quite a lot of technical work is needed, e.g.:**

- to handle tricky boundary conditions (Dirichlet for $\int \mathrm{Re}[f^2dz]$);
- to prove convergence, incl. near singularities [ complex analysis ];
- to recover the normalization of $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1}\ldots\sigma_{u_n}]$ [ probability ].

## Explicit formulae for spin correlations in the general case

We *define* $\langle\boldsymbol{\sigma}_{u_1}\ldots\boldsymbol{\sigma}_{u_n}\rangle_{\boldsymbol{\Omega}}^{+} := \exp[\,\int\mathcal{L}(u_1,\ldots,u_n)\,]$, where

$$\mathcal{L}_\Omega(u_1,\ldots,u_n) := \textstyle\sum_{s=1}^{n}\mathrm{Re}\,[\,\mathcal{A}_\Omega(u_s;u_1,...,\hat{u}_s,...,u_n)du_s\,],$$

and the multiplicative normalization is chosen so that

$$\begin{array}{rcll}\langle\sigma_{u_1}...\sigma_{u_n}\rangle_\Omega^+ & \sim & \langle\sigma_{u_1}....\sigma_{u_{n-1}}\rangle_\Omega^+\cdot\langle\sigma_{u_n}\rangle_\Omega^+ & \text{as }\; u_n\to\partial\Omega\,,\\ \langle\sigma_{u_1}\sigma_{u_2}\rangle_\Omega^+ & \sim & |u_2-u_1|^{-\frac{1}{4}} & \text{as }\; u_2\to u_1\in\Omega\,.\end{array}$$

Coefficients $\boldsymbol{\mathcal{A}_\Omega(u_1;u_2,...,u_n)}$ are *defined* via the following b.v.p.:

○ $f(z^\sharp)\equiv -f(z^\flat)$ is a holomorphic spinor on $[\Omega;u_1,...,u_n]$;

○ $\mathrm{Im}\big[\,f(\zeta)(n(\zeta))^{\frac{1}{2}}\,\big]=0$ for $\zeta\in\partial\Omega$;

○ $f(z)=ic_s\cdot(z-u_s)^{-\frac{1}{2}}+\ldots$ for some (unknown) $c_s\in\mathbb{R}$, $s\geqslant 2$;

○ $f(z)=(z-u_1)^{-\frac{1}{2}}+2\mathcal{A}_\Omega(u_1;u_2,...,u_n)\cdot(z-u_1)^{\frac{1}{2}}+\ldots$

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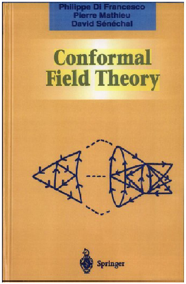
**Remarks:** • The closeness of the differential form $\mathcal{L}_{\Omega,n}$ and the existence of an appropriate multiplicative normalization are not immediate (can be deduced along the proof of convergence);

• Similar techniques can be applied for more involved boundary conditions and/or in the multiply connected setup (when no explicit formulae are available), as well as to other fields.

## Some research routes / open questions

• Better understanding of the CFT description at criticality: other fields, fusion rules, height functions, "geometric" observables (e.g., probabilities of concrete topologies of domain walls)



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• Near-critical (massive) regime $x - x_{\mathrm{crit}} = m \cdot \delta$: convergence of correlations, massive $\mathrm{SLE}_3$ curves and loop ensembles etc.

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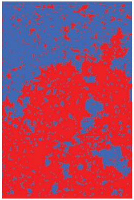
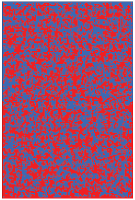


$x = x_{\mathrm{crit}}$

• **Renormalization**

$$\text{fixed } x > x_{\mathrm{crit}},\ \delta \to 0 \quad \xrightarrow{\hspace{3cm}} \quad (x - x_{\mathrm{crit}}) \cdot \delta^{-1} \to \infty$$



$x = 1$

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Tool: local relations and spinor observables are always there!

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Extended version of this talk: `arXiv:1605.09035`

# Thank you!