# Small Bergman-Orlicz spaces and their composition operators

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# Definitions

Orlicz function: a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ convex such that $\psi(x)/x \to \infty$ and $\psi(0) = 0$.

## Orlicz spaces

Let $(\Omega, \mathbb{P})$ be a probability space and $\psi$ an Orlicz function. The Orlicz space $L^\psi = L^\psi(\Omega, \mathbb{P})$ is:

$$\left\{ f : \Omega \to \mathbb{C} \text{ measurable, } \exists \alpha > 0, \int_\Omega \psi\left(\frac{|f|}{\alpha}\right) d\mathbb{P} < \infty \right\}.$$

Morse-Transue space $M^\psi = M^\psi(\Omega, \mathbb{P})$ is:

$$\left\{ f : \Omega \to \mathbb{C} \text{ measurable, } \forall \alpha > 0, \int_\Omega \psi\left(\frac{|f|}{\alpha}\right) d\mathbb{P} < \infty \right\}.$$

Endowed with the Luxemburg norm

$$\|f\|_\psi := \inf\{\alpha > 0, \int_\Omega \psi\left(\frac{|f|}{\alpha}\right) d\mathbb{P} \le 1\},$$

$L^\psi$ and $M^\psi$ are Banach spaces and $M^\psi = \mathrm{clos}(L^\infty)$ in $L^\psi$.

### $L^p$ spaces

If $\psi(x) = x^p$ then $L^\psi = M^\psi = L^p$. More generally $L^\psi = M^\psi$ if and only if $\psi$ satisfies the $\Delta_2$-*condition*.

## $A^{\psi}_{\alpha}$ and $H^{\psi}$, $N \geq 1$.

1. The weighted Bergman-Orlicz space $A^{\psi}_{\alpha}$ is the set $L^{\psi}(\mathbb{B}_N, v_{\alpha}) \cap H(\mathbb{B}_N)$, with $v_{\alpha} = c_{\alpha}(1 - |z|^2)^{\alpha} v$, $\alpha > -1$; we also let $AM^{\psi}_{\alpha} := M^{\psi}(\mathbb{B}_N, v_{\alpha}) \cap H(\mathbb{B}_N)$.
2. The Hardy-Orlicz space $H^{\psi}$ is defined as

$$\left\{ f \in H(\mathbb{B}_N);\ \sup_{0<r<1} \|f_r\|_{L^{\psi}(\mathbb{S}_N, \sigma_N)} < \infty \right\},$$

and $HM^{\psi} = M^{\psi}(\mathbb{S}_N, \sigma_N) \cap H(\mathbb{B}_N)$.

1. $A^{\psi}_{\alpha}$, $AM^{\psi}_{\alpha}$ (resp. $H^{\psi}$ and $HM^{\psi}$) are Banach subspaces of $L^{\psi}(\mathbb{B}_N, v_{\alpha})$ (resp. $L^{\psi}(\mathbb{S}_N, \sigma_N)$);
2. If $\psi$ satisfies the $\nabla_2$-condition then $(AM^{\psi}_{\alpha})^{**} = A^{\psi}_{\alpha}$ (resp. $(HM^{\psi})^{**} = H^{\psi}$).

# A simple observation

## Evaluations $\alpha \geq -1$

Point evalution functionals $\delta_z$ are continuous on $A_\alpha^\psi$:

$$\|\delta_z\| \approx \psi^{-1}\left(\frac{1}{(1-|z|)^{N(\alpha)}}\right).$$

Therefore

$$A_\alpha^\psi \subset H_v^\infty := \left\{f \in H(\mathbb{B}_N),\ \|f\|_v := \sup_z \frac{|f(z)|}{v(z)} < \infty\right\},$$

where $v(z) = \psi^{-1}\left(\frac{1}{(1-|z|)^{N(\alpha)}}\right)$.

## Observation

If $\alpha > -1$ and $\psi$ grows fast (namely satisfies the $\Delta^2$-*condition*), then $z \mapsto \|\delta_z\| \in L^\psi(\mathbb{B}_N, v_\alpha)$.

# A simple observation

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$$A^\psi_\alpha \subset H^\infty_v := \left\{ f \in H(\mathbb{B}_N),\ \|f\|_v := \sup_z \frac{|f(z)|}{v(z)} < \infty \right\},$$

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## Observation

If $\alpha > -1$ and $\psi$ grows fast (namely satisfies the $\Delta^2$-*condition*), then $z \mapsto \|\delta_z\| \in L^\psi(\mathbb{B}_N, v_\alpha)$.

**Direct consequences** $\alpha > -1$

Assume that $\psi \in \Delta^2$.

1. $A^\psi_\alpha = H^\infty_v$ with equivalent norms, where $v(z) = \psi^{-1}\left(\frac{1}{1-|z|}\right)$;
2. $AM^\psi_\alpha = H^0_v$ where $v(z) = \psi^{-1}\left(\frac{1}{1-|z|}\right)$;
3. $A^\psi_\alpha = A^\psi_\beta$ for any $\beta > -1$;
4. Every bounded operator from $A^\psi_\alpha$ into $L^\psi$ is order bounded into $L^\psi$.

Recall that an operator $T$ from a Banach space $X$ to a Banach lattice $Y$ is order bounded into $Z$, $Z$ a sublattice of $Y$, if there exists $y \in Z$ such that for every $x \in X$, $\|x\| \le 1$, $|Tx| \le y$.

Let $\psi \in \Delta^2$ and $\alpha > -1$. For any $\phi : \mathbb{B}_N \to \mathbb{B}_N$ holomorphic, the composition operator $C_\phi : f \mapsto f \circ \phi$, $f \in A^\psi_\alpha$, is order bounded into $L^\psi$.

**Theorem 1**

*TFAE:*

*1. Every composition operator is bounded on $A^\psi_\alpha(\mathbb{B}_N)$;*

*2. Every composition operator is bounded on $AM^\psi_\alpha(\mathbb{B}_N)$;*

*3. Every composition operator acting on $A^\psi_\alpha(\mathbb{B}_N)$ is order bounded into $L^\psi(\mathbb{B}_N, v_\alpha)$;*

*4. $A^\psi_\alpha(\mathbb{B}_N) = H^\infty_v$ with $v(z) = \psi^{-1}\left(\frac{1}{1-|z|}\right)$;*

*5. $\psi$ satisfies the $\Delta^2$-condition.*

# Applications to composition operators

Let $\psi \in \Delta^2$ and $\alpha > -1$. For any $\phi : \mathbb{B}_N \to \mathbb{B}_N$ holomorphic, the composition operator $C_\phi : f \mapsto f \circ \phi$, $f \in A^\psi_\alpha$, is order bounded into $L^\psi$.

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*4. $A^\psi_\alpha(\mathbb{B}_N) = H^\infty_v$ with $v(z) = \psi^{-1}\left(\frac{1}{1-|z|}\right)$;*

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# Applications to composition operators - 2

**Theorem 2**

*Let $\alpha \geq -1$, let $\psi$ be an Orlicz function satisfying the $\Delta^2$-condition and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. TFAE:*
*1. $C_\phi$ is compact on $A^\psi_\alpha(\mathbb{B}_N)$;*
*2. $C_\phi$ acting on $A^\psi_\alpha(\mathbb{B}_N)$ is order bounded into $M^\psi(\mathbb{B}_N)$;*
*When $\alpha > -1$ we have the following*
*3.*

$$\left\| C_\phi \right\|_e = \lim_{|z|\to 1} \frac{\psi^{-1}\left(1/(1-|\phi(z)|)\right)}{\psi^{-1}\left(1/(1-|z|)\right)}.$$

# Between $\Delta_2$ and $\Delta^2$?

There exists $\psi \in \Delta^1$ such that "boundedness", "order boundedness into $L^\psi$", "order boundedness into $M^\psi$" and "compactness" are all distinguished by composition operators.

# Between $\Delta_2$ and $\Delta^2$?

$$C_\alpha(h) = \begin{cases} \{z \in \mathbb{B}_N,\ 1-|z|<h\} & \text{if } \alpha > -1 \\ \{z \in \overline{\mathbb{B}_N},\ 1-|z|<h\} & \text{if } \alpha = -1 \end{cases}.$$

## $\Delta^1$-condition

Let $\alpha \geq -1$, $\psi \in \Delta^1$ and $\phi : \mathbb{B}_N \to \mathbb{B}_N$ holomorphic.

1. $C_\phi$ acting on $A^\psi_\alpha(\mathbb{B}_N)$ is order bounded into $L^\psi(\mathbb{B}_N)$ iff $\exists A > 0$ such that $\forall h \in (0,1)$,

$$\mu_\phi(C_\alpha(h)) \leq \frac{1}{\psi(A\psi^{-1}(1/h^{N(\alpha)}))}.$$

2. $C_\phi$ acting on $A^\psi_\alpha(\mathbb{B}_N)$ is order bounded into $M^\psi(\mathbb{B}_N)$ iff $\forall A > 0$, $\exists C_A > 0$ and $\exists h_A \in (0,1)$ such that $\forall h \in (0, h_A)$,

$$\mu_\phi(C_\alpha(h)) \leq \frac{C_A}{\psi(A\psi^{-1}(1/h^{N(\alpha)}))}.$$