Hardy-Hodge decomposition of vector fields

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honoring the memory of V.P. Havin

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- For $n \geq 3$, we consider harmonic potentials in divergence form:

$$P_{\mathrm{div}V}(x) = \int \frac{x-y}{|x-y|^{n-2}} \mathrm{div}\, V(y)$$

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- They occur frequently when modeling electro-magnetic phenomena in the quasi-static approximation to Maxwell's equations.

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- Likewise, the inverse magnetization problem is to recover the magnetization $\mathbf{M}$ on a given object, from measurements of the field $H = -\nabla \phi$ near the object.
- Today, inverse magnetization problems are a hot topic in Earth and Planetary Sciences.

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- Let us look at the elementary case where V is supported on the horizontal plane with L^p density there, $1 < p < \infty$.
- This geometry is in fact realistic in scanning microscopy of rocks which are typically sanded down to thin slabs.

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$$R_j(f)(Y) := \lim_{\epsilon \to 0} \frac{1}{2\pi} \iint_{\mathbb{R}^2 \setminus B(Y,\epsilon)} f(X') \frac{(y_j - x_j')}{|Y - X'|^3} \, dX', \qquad j = 1, 2,$$

are the Riesz transforms.

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- M is silent (from both sides) iff $R_1 m_1 + R_2 m_2 = 0$ and $m_3 = 0$.

Question

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To approach it, we introduce some classical function spaces.

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- Let $\mathfrak{H}^p_+$ consist of ∇u, u harmonic in $\{x_3 > 0\}$, such that

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- We put $\mathcal{D}^p$ for divergence-free vector fields in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.

The Hardy-Hodge decomposition on $\mathbb{R}^2$

Theorem (L.B., D. Hardin, E. Lima, E.B. Saff, B. Weiss)

For $1 < p < \infty$ *one has the orthogonal sum:*

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where $\mathcal{G}^p$ is the space of distributional gradients in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.

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- Easily checked using $R_1^2 + R_2^2 = -\mathrm{Id}$.

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- M is silent iff it is tangent and divergence-free.
- Transparent if we observe the orthogonality:

$$\mathfrak{H}_+^p \perp \mathfrak{H}_-^q \quad \text{and} \quad \mathcal{D}^p \times \{0\} \perp \mathfrak{H}_\pm^q, \quad 1/p + 1/q = 1.$$

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- It extends to any class of functions or of distributions invariant under Riesz transforms, *e.g.* $\mathfrak{h}^1$, BMO, $W^{-\infty,p}$ (*i.e.* finite sums of derivatives of any order of L^p-functions, $1 < p < \infty$).

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- If $M \in (L^2(\mathbb{R}^n))^3$ then $P_{\mathfrak{H}^2_-} M$ yields the magnetization of least $(L^2(\mathbb{R}^n))^3$-norm which is equivalent to M from above.

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- One can then define $\mathcal{D}^p = (\mathcal{G}^q)^{\perp}$ for the pairing

$$(G, D) := \int_{\mathcal{M}} \langle G, D \rangle_{\mathcal{M}} d\sigma, \quad 1/p + 1/q = 1.$$

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- When $\mathcal{M}$ is smooth the decomposition holds in more general spaces of functions or distributional currents.

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- Thus, we are left to decompose $V - D - \nabla u$ which is a normal vector field on $\mathcal{M}$. For this we need preliminaries in Clifford analysis. We restrict to $n = 3$ for simplicity.

Some Clifford analysis

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- $\mathfrak{C}$ is the skew unital algebra generated over $\mathbb{R}$ by $\{e_1, e_2, e_3\}$ with:

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- The norm of z is $|z| = (\sum_{0 \le k \le 3} x_k^2 + \sum_{i<j} x_{i,j}^2 + x_{123}^2)^{1/2}$.

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Lemma

A vector-valued function is left monogenic if and only if it is monogenic, if and only if it is the gradient of a harmonic function.

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- If f is left monogenic in Ω^+ and its nontangential maximal function lies in $L^p(\mathcal{M})$, then f has a nontangential limit $f^+ \in L^p(\mathcal{M})$ a.e. on $\mathcal{M}$ (Verchota), and by the Green formula (see *e.g.* "Clifford Algebras and Dirac Operators in Analysis" by Gilbert and Murray):

$$f(z) = \mathcal{C}f^+(z) := \frac{1}{4\pi}\int_{\mathcal{M}} \frac{\overline{y-z}}{|y-z|^3} n(y) f^+(y) d\sigma(y), \qquad z \in \Omega^+.$$

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$$\mathcal{SC}h(y) = \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{\mathcal{M}\setminus B(y,\varepsilon)} \frac{\overline{\xi - y}}{|\xi - y|^3} n(\xi)h(\xi)d\sigma(\xi), \qquad y \in \mathcal{M}.$$

- This gives us an analog of the Plemelj formula:

$$\mathcal{C}^{+}h(y) - \mathcal{C}^{-}h(y) = h(y).$$

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- Write $h = \mathcal{C}^+ h - \mathcal{C}^- h$ by Plemelj formula. Since h is normal, $\mathcal{C}^{\pm} h \in \mathcal{H}^p_{\pm}$. Indeed, if $h(y)$ is normal to $\mathcal{M}$ at y, the $\mathfrak{C}$-product $n(y)h(y)$ is scalar-valued, so the integrand in the definition of $\mathcal{C}h$ is vector valued.

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- Uniqueness follows from uniqueness of the Hodge decomposition and the Liouville theorem for harmonic functions.

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Proposition

The distribution is silent from outside if and only if

$$2\pi\psi(y) = -\lim_{\varepsilon\to 0}\int_{\mathcal{M}\setminus B(y,\varepsilon)} \frac{\xi - y}{|\xi - y|^3}.(\psi n + R^{-1}(D))(\xi)d\sigma(\xi) \quad y \in \mathcal{M}.$$

More complicated equation involving the curvature.

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More complicated equation involving the curvature. When the latter is constant one gets the previous characterization. Still silence from both sides means tangent and divergence-free.

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- In $\mathbb{R}^n$, let rational approximation mean approximation by gradients of discrete harmonic potentials with finitely many masses. The Hardy-Hodge decomposition implies:

Theorem

Let S be a Lipschitz regular surface patch on a compact connected smooth hypersurface $\mathcal{M} \subset \mathbb{R}^n$. Let v be $\mathbb{R}^n$-valued in $L^p(S)$, $1 < p < \infty$. Then, v can be approximated arbitrarily close by rationals in $L^p(S)$ iff the tangential component of v is a gradient.

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More "rational" approximation

- By [V.P. Havin-S. Smirnov, 1999], a measure with compact support containing no simple rectifiable arc of positive length cannot have a distributional divergence which is again a measure. The same holds on a smooth manifold.
- A fortiori then, a compact set containing no such arc is a grad-set for L^p (any field is approximable y a gradient). The Hardy-Hodge decomposition now implies:

Theorem

Let K be a closed set in a compact connected smooth hypersurface $\mathcal{M} \subset \mathbb{R}^n$, and assume that K contains no simple rectifiable arc of positive length. Then, each $\mathbb{R}^n$-valued v in $L^p(K)$ can be approximated arbitrary close by rationals in $L^p(K)$, $1 < p < \infty$.