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Multiple operator integrals in perturbation theory

Abstract

Double operator integrals appeared in a paper by Yu.L. Daletskii and S.G. Krein who observed that such integrals arise in a natural way in perturbation theory. Later M.S. Birman and M.Z. Solomyak developed a beautiful theory of double operator integrals. Double operator integrals are expressions of the form

$$\iint \Phi(x_1,x_1)\,dE_1(x_1)T\,dE_2(x_2),$$

where E_1 and E_2 are spectral measures, T is a bounded operator on Hilbert space and Φ is a measurable function satisfying certain assumptions. Functions satisfying such assumptions are called *Schur multipliers*.

Birman and Solomyak observed that if f is a function on $\mathbb{R}$ such that the divided difference $\mathfrak{D}f$, $(\mathfrak{D}f)(s,t) \stackrel{\mathrm{def}}{=} (f(s)-s(t))(s-t)^{-1}$, is a Schur multiplier, then for arbitrary self-adjoint operators A and B with bounded $A-B$, the following formula holds:

$$f(A)-f(B)=\iint(\mathfrak{D}f)(s,t)\,dE_A(s)(A-B)\,dE_B(t),$$

where E_A and E_B are the spectral measures of A and B. This implies that

$$\|f(A)-f(B)\|\le\mathrm{const}\,\|A-B\|.$$

Functions satisfying this inequality are called *operator Lipschitz*. It turns out that the converse is also true: if f is operator Lipschitz, then $\mathfrak{D}f$ is a Schur multiplier.

I am going to give sufficient conditions and necessary conditions for operator Lipschitzness.

Then I will introduce the notion of operator Hölder functions of order α, $0<\alpha<1$. It turns out that unlike in the case of operator Lipschitz functions, the class of operator Hölder functions of order α coincides with the class of Hölder functions of order α.

I am also going to consider the problem of estimating the norms of $f(A)-f(B)$ in Schatten–von Neumann classes.

The situation for functions of normal operators and for functions of n-tuples of commuting self-adjoint operators is more complicated. However, it turns out that the results mentioned above can be generalized.

In the final lecture I am going to speak about my recent joint results with A.B. Aleksandrov and F.L. Nazarov. I will introduce functions $f(A,B)$ of noncommuting

self-adjoint operators A and B. This problem leads naturally to triple operator integrals:

$$f(A_1,B_1)-f(A_2,B_2)=\iiint\big(\mathfrak{D}^{[1]}f\big)(x_1,x_2,y)\,dE_{A_1}(x_1)(A_1-A_2)\,dE_{A_2}(x_2)\,dE_{B_1}(y)$$
$$+\iiint\big(\mathfrak{D}^{[2]}f\big)(x,y_1,y_2)\,dE_{A_2}(x)\,dE_{B_1}(y_1)(B_1-B_2)\,dE_{B_2}(y_2),$$

where the divided differences $\mathfrak{D}^{[1]}f$ and $\mathfrak{D}^{[2]}f$ are defined by

$$\mathfrak{D}^{[1]}f(x_1,x_2,y)\stackrel{\mathrm{def}}{=}\frac{f(x_1,y)-f(x_2,y)}{x_1-x_2}\quad\text{and}\quad\mathfrak{D}^{[2]}f(x,y_1,y_2)\stackrel{\mathrm{def}}{=}\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2},$$

To justify this formula we define Haagerup-like tensor products of the first kind and of the second kind and define triple operator integral for functions in such Haagerup-like tensor products. Then we prove that if f is a function on $\mathbb{R}^2$ of Besov class $B^1_{\infty,1}(\mathbb{R}^2)$, then $\mathfrak{D}^{[1]}f$ belongs to the Haagerup tensor product of the first kind, while $\mathfrak{D}^{[2]}f$ belongs to the Haagerup tensor product of the second kind. This implies that if $1<p<2$, $f\in B^1_{\infty,1}(\mathbb{R}^2)$, then the following Lipschitz type estimate in the Schatten–von Neumann norm $\boldsymbol{S}_p$ holds:

$$\|f(A_1,B_1)-f(A_2,B_2)\|_{\boldsymbol{S}_p}\le\mathrm{const}\max\{\|A_1-A_2\|_{\boldsymbol{S}_p},\|A_1-A_2\|_{\boldsymbol{S}_p}\}.$$

On the other hand, it turns out that there is no such a Lipschitz estimate in $\boldsymbol{S}_p$ for $p>2$ as well as in the operator norm.